Lecture 1 - BDM_MATHSV
Bases on Sets theory and Enumeration

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- 6 lectures
- 6 tutorial sessions
- EVC (0.3) quiz during lectures
- EVI (0.7) final examination: 5th November 2025 (S1 or S2 ?)

Course content available on Hippocampus
https://hippocampus.ec-nantes.fr/course/view.php?id=2883

Timetable

CM	TD BBA	TD BSC
16/09 S1	16/09 S2	17/09 S1
23/09 S1	23/09 S2	24/09 S1
30/09 S1	30/09 S2	01/10 S1
07/10 S1	07/10 S2	08/10 S1
14/10 S1	14/10 S2	15/10 S1
22/10 S1	21/10 S1	15/10 S2

- Not yet on Onboard
- CM after TD

Course Outline

1. Bases on Sets theory and Enumeration
2. Introduction to probabilities : from events to random variables
3. Discrete and continuous random variables
4. Convergence of random variables and Limit theorems
5. Estimators in Statistics and Confidence intervals
6. Statistical tests

Lecture Outline

1. Empty set, complementary set
2. Operations on sets: union, intersection, product; De Morgan's law...
3. Subsets, partition
4. Cardinality, Addition rule
5. Enumeration: permutation, partial permutation, p-list, combination
6. Binomial coefficient: Pascal's triangle, Newton's binomial formula

Review of Sets

Review of Sets

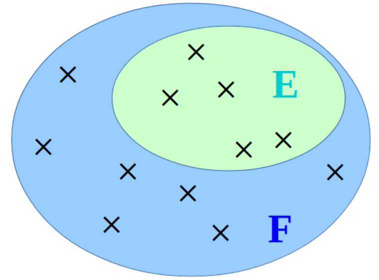
A set E is a collection of distinct elements. ☞ Examples :

- $E = \{\} = \varnothing$.
- $E = \{0, 1\}$.
- $E = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}, \mathrm{y}\}$.
- $E = \{{\color{green}\bullet}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{yellow}\bullet}, \bullet, {\color{magenta}\bullet}\}$.
- $E = \{0, 1, 2, 3, \ldots\} = \mathbb{N}$.
- $E = \mathbb{R}$.
- $E = \{\text{points inside the circle with center O and radius } 1\}$.

Review of Sets

A set E is included in a set F if all elements of E are also in F:

$$E \subset F \iff \forall x \in E, \; x \in F.$$



Review of Sets

Two sets E and F are equal if they contain the same elements:

$$E = F \iff E \subset F \text{ and } F \subset E.$$

This is called double inclusion.

☞ Examples :

- $\{0, 1\} = \{1, 0\}$.
- $\mathbb{N} \subset \mathbb{Z}$ but $\mathbb{N} \neq \mathbb{Z}$ because $\mathbb{Z} \not\subset \mathbb{N}$.
- $\{\textcolor{green}{\bullet}, \textcolor{blue}{\bullet}\} \subset \{\textcolor{green}{\bullet}, \textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{yellow}{\bullet}, \textcolor{black}{\bullet}, \textcolor{magenta}{\bullet}\}$.
- As soon as E has one or more elements, $\varnothing \subset E$ but $E \not\subset \varnothing$.

Review of Sets

The cardinality of a set E is the number of elements it contains.

☞ Examples :

- $E = \varnothing$: $\mathrm{Card}(E) = 0$.
- $E = \{0, 1\}$: $\mathrm{Card}(E) = 2$.
- $E = \{{\color{green}\bullet}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{yellow}\bullet}, \bullet, {\color{magenta}\bullet}\}$: $\mathrm{Card}(E) = 6$.
- $E = \mathbb{N}$: $\mathrm{Card}(E) = +\infty$.
- $E = [0\,;1]$: $\mathrm{Card}(E) = +\infty$.

☞ Special cases: singleton $\{x\}$, pair $\{x, y\}$, triplet $\{x, y, z\}$.

Review of Sets

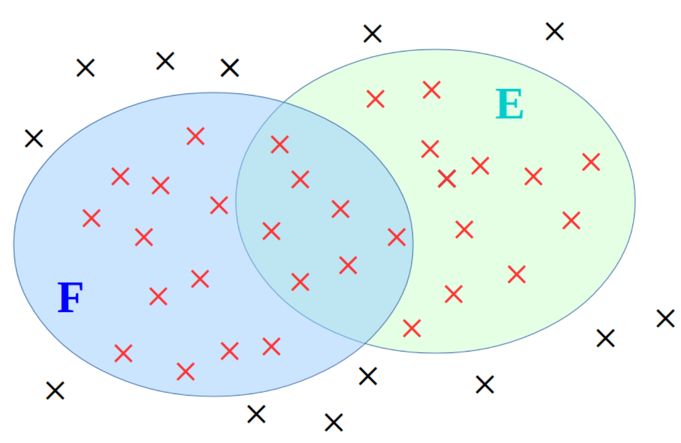
The union of two sets E and F consists of elements belonging to either E or F:

$$x \in E \cup F \iff x \in E \ \underline{\underline{\text{or}}} \ x \in F.$$

☞ Examples :

- $\{0\} \cup \{1\} = \{0, 1\}$.
- $\{\mathrm{i}, \mathrm{o}, \mathrm{y}\} \cup \{\mathrm{e}, \mathrm{i}, \mathrm{y}\} = \{\mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{y}\}$.
- $[0\,;1] \cup [\frac{1}{2}\,;\frac{3}{2}] = [0\,;\frac{3}{2}]$.
- $\{\textcolor{red}{\bullet}, \textcolor{yellow}{\bullet}, \textcolor{green}{\bullet}, \textcolor{blue}{\bullet}\} \cup \{\bullet, \textcolor{red}{\bullet}, \textcolor{blue}{\bullet}\} = \{\textcolor{red}{\bullet}, \textcolor{yellow}{\bullet}, \textcolor{green}{\bullet}, \textcolor{blue}{\bullet}, \bullet\}$.

$E \cup F$



Review of Sets

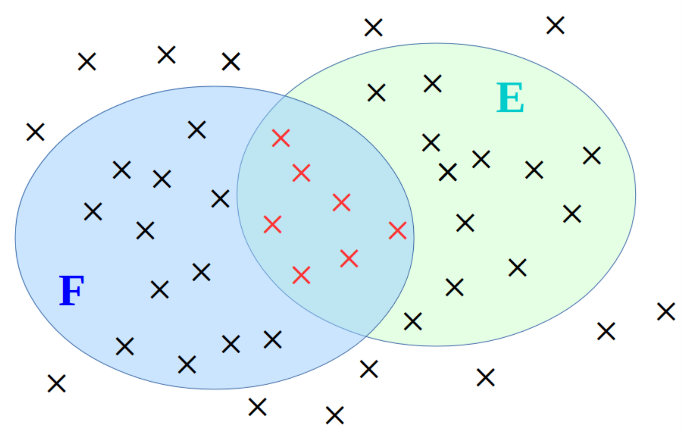
The intersection of two sets E and F consists of elements belonging to both E and F:

$$x \in E \cap F \iff x \in E \ \underline{\underline{\text{and}}} \ x \in F.$$

☞ Examples :

- $\{0\} \cap \{1\} = \varnothing$.
- $\{0, 1\} \cap \{1\} = \{1\}$.
- $\{\mathrm{i}, \mathrm{o}, \mathrm{y}\} \cap \{\mathrm{e}, \mathrm{i}, \mathrm{y}\} = \{\mathrm{i}, \mathrm{y}\}$.
- $[0\,;1] \cap [\frac{1}{2}\,;\frac{3}{2}] = [\frac{1}{2}\,;1]$.
- $\{{\color{red}\bullet}, {\color{yellow}\bullet}, {\color{green}\bullet}, {\color{blue}\bullet}\} \cap \{\bullet, {\color{red}\bullet}, {\color{blue}\bullet}\} = \{{\color{red}\bullet}, {\color{blue}\bullet}\}$.

$E \cap F$



Union and Intersection of multiple sets

Given m sets $A_1, A_2, \dots A_m$:

- their union is
  $\bigcup_{j=1}^{m} A_j = A_1 \cup \dots \cup A_m = \{x \in \Omega \mid \exists j \in \{1, \dots, m\}, \, x \in A_j\}$,
- their intersection is
  $\bigcap_{j=1}^{m} A_j = A_1 \cap \dots \cap A_m = \{x \in \Omega \mid \forall j \in \{1, \dots, m\}, \, x \in A_j\}$.

The symbol $\exists$ means it exists and $\forall$ means for all.

Review of Sets

The complement of a set E consists of elements not belonging to this set:

$$x \in \bar{E} \iff x \notin E.$$

The difference between F and E contains the elements of F that do not belong to E:

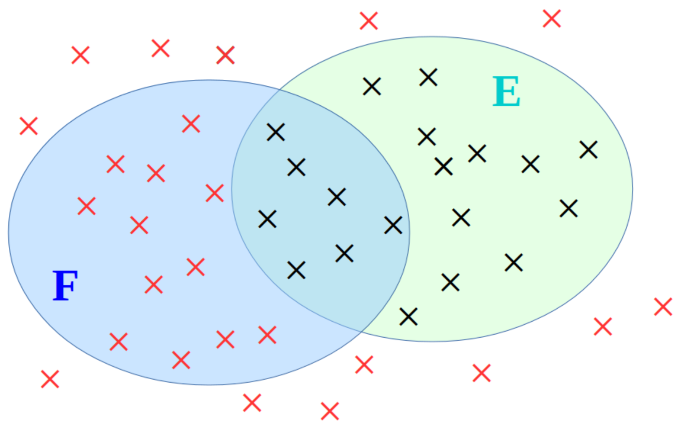
$$x \in F \backslash E \iff x \in F \ \underline{\underline{\text{but}}} \ x \notin E.$$

☞ Examples :

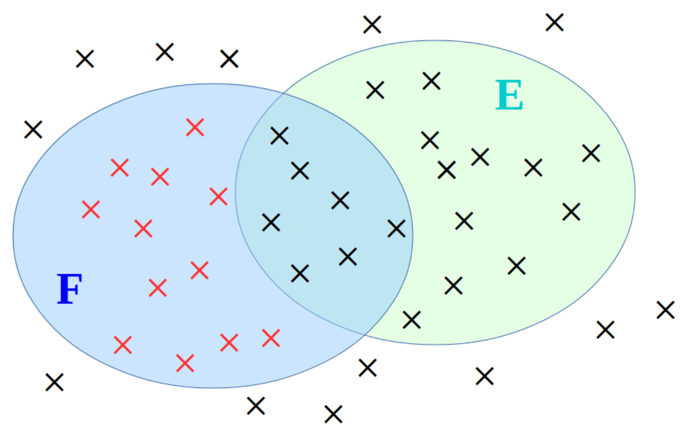
- In $\{0, 1\}$, $\overline{\{0\}} = \{1\}$.
- In $\{a, e, i, o, u, y\}$, $\overline{\{e, o, u, y\}} = \{a, i\}$.
- $[0\,;1] \backslash [\frac{1}{4}\,; \frac{3}{4}] = [0\,; \frac{1}{4}) \cup (\frac{3}{4}\,; 1]$.
- $\{\textcolor{green}{\bullet}, \textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{yellow}{\bullet}, \bullet, \textcolor{magenta}{\bullet}\} \backslash \{\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}\} = \{\textcolor{green}{\bullet}, \textcolor{yellow}{\bullet}, \bullet, \textcolor{magenta}{\bullet}\}$.

Review of Sets

$\bar{E}$



$F \setminus E$



Review of Sets

☞ Some useful relations: For every sets A, B and C:

- Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
- Idempotence: $A \cap A = A$, $A \cup A = A$
- Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Double negation: $\overline{(\overline{A})} = A$
- Excluded third: $A \cup \overline{A} = \Omega$, $A \cap \overline{A} = \emptyset$
- Absorption: $A \cup \Omega = \Omega$, $A \cup (A \cap B) = A$, $A \cap \emptyset = \emptyset$, $A \cap (A \cup B) = A$
- Neutral element: $A \cup \emptyset = A$, $A \cap \Omega = A$
- De Morgan's law: $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$, $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

☞ For practice: highlight these formulas in diagrams.

Product set

The product $A \times B$ of two sets A and B is the set of pairs of two elements with the first one in A and the second one in B:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

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The product $A_1 \times \ldots \times A_n$ of n sets $A_1, \ldots A_n$ is the set of tuples which i-th coordinate is in A_i for every $i \in \{1, \ldots, n\}$:

$$A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i, \forall i \in \{1, \ldots, n\}\}.$$

When $A_1 = A_2 = \ldots = A_n$, the product $A_1 \times \ldots \times A_n$ is denoted by A^n.

Example of products of sets

☞ Example :

- The outcomes of the experiment of rolling 2 dice successively are in the set $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} = \{1,2,3,4,5,6\}^2$. The subset of outcomes "The first die gives an odd number" is: $A = \{1,3,5\} \times \{1,2,3,4,5,6\}$. Examples of elements in A are: $(1,2)$, $(1,1)$, $(5,6)$, etc.
- $\{1,2\} \times \{3\} = \{(1,3),(2,3)\}$
- $\{1,2\} \times \{2,3\} = \{(1,2),(1,3),(2,2),(2,3)\}$
- $\{1,2\} \times \emptyset = \emptyset$
- $\{3\} \times \{1,2\} = \{(3,1),(3,2)\} \neq \{1,2\} \times \{3\}$.

Partition

A partition $\{A_1, \ldots, A_m\}$ of a set Ω is a family of m sets satisfying:

- The sets are disjoint and nonempty: $\forall i, j \in \{1, \ldots, m\}$, $A_i \cap A_j = \emptyset$,
- The sets cover Ω: $\bigcup_{j=1}^{m} A_j = \Omega$.

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☞ Examples :

- $\{A, \overline{A}\}$ is a partition of Ω, provided that $A \neq \emptyset$ and $A \neq \Omega$.
- If A and B are two distinct sets and so that $A \neq \overline{B}$, different from $\emptyset$ and Ω, then $\{A \cap B, A \cap \overline{B}, \overline{A} \cap B, \overline{A} \cap \overline{B}\}$ is a partition of Ω.

Particular partition

- If $A_1, A_2, \ldots, A_m$ are mutually disjoint, then $Card(A_1 \cup \ldots \cup A_m) = Card(A_1) + \ldots + Card(A_m)$.
- If $A_1, A_2, \ldots, A_m$ is a partition of Ω, then $Card(\Omega) = Card(A_1) + \ldots + Card(A_m)$.

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☞ Example : In a class of 23 students, we consider the following sets: A the set of students that practice sport, B the set of students that wear glasses. If 9 students with glasses and 5 students without glasses practice sport, 6 boys without glasses do not practice sport, then the number of students with glasses who practice sport is:

$$\begin{aligned} Card(\overline{A} \cap B) &= Card(\Omega) - Card(A \cap B) - Card(A \cap \overline{B}) - Card(\overline{A} \cap \overline{B}) \\ &= 23 - 9 - 5 - 6 \end{aligned}$$

since $\{A \cap B, \overline{A} \cap B, A \cap \overline{B}, \overline{A} \cap \overline{B}\}$ is a partition of Ω.

A Bit of Combinatorics

A Bit of Combinatorics

Doing enumeration or combinatorics means knowing how to count the possible outcomes of an experiment. ☞ Examples :

- There are 2 outcomes for a coin toss.
- There are 37 outcomes for a roulette spin.
- There are 120 anagrams of the word TABLE.
- There are 10,000 different PIN codes.
- There are 201,376 hands of 5 cards in a 32-card deck.
- There are 24,165,120 ordered hands of 5 cards in the same deck.

A Bit of Combinatorics

How many different PIN codes are there?

☞ There are $10 \times 10 \times 10 \times 10 = 10^4$. How many different PIN codes are there with distinct digits?

☞ There are $10 \times 9 \times 8 \times 7 = 5040$.

p-lists

A p-list $(x_1, \ldots, x_p)$ is a tuple of p elements in a set Ω of n elements. That is, an element of Ω^p.

The cardinality of Ω^p, the set of p-lists in Ω, is $Card(\Omega^p) = n^p$, if $Card(\Omega) = n$.

☞ Examples :

- A phone number in France is $+33$ followed by 9 numbers in $\{0, 1, \ldots, 9\}$. It can be seens as a 9-list of $\Omega = \{0, 1, \ldots, 9\}$. Thus, the number of phone numbers is $Card(\{0, 1, \ldots, 9\}^9) = 10^9 = 1000000000$.
- There are $2^5 = 32$ possible results for tossing 5 coins successively. For instance, (H, H, T, H, T) is one of them.

A Bit of Combinatorics

The number of arrangements of p elements chosen from n elements represents the number of ordered lists of size p that can be constructed from the n elements. We have:

$$A_n^p = n \times (n-1) \times \ldots \times (n-p+1) = \frac{n\,!}{(n-p)\,!}.$$

If $p = n$, we speak of permutations. There are $n\,!$ for a set of n elements.

☞ Example : TABLE, TABEL, TALBE, ..., ELBAT. $5\,! = 120$ in total.

Anagrams

If n elements can be divided into c classes of alike elements, differing from class to class, then the number of permutations of these elements taken all at a time is

$$\frac{n!}{n_1!n_2!\ldots n_c!}$$

(with $n = n_1 + n_2 + \ldots + n_c$ and n_j is the number of elements in the j-th class).

☞ Example : There are 5! anagrams of the word "maths", $\frac{11!}{2!\times 2!\times 2!}$ of the word "mathematics" (2 "a", 2 "m", 2 "t") and $\frac{6!}{2!\times 3!}$ of "baobab" (2 "a", 3 "b").

Combinations

What if we do not consider order? The number of combinations of p elements chosen from n elements represents the number of sets of p elements that can be constructed from the n elements. We have:

$$C_n^p = \binom{n}{p} = \frac{n\,!}{p\,!\,(n-p)\,!} = \frac{A_n^p}{p\,!}.$$

Indeed, there are A_n^p ordered lists of p elements. If we do not consider order, the $p\,!$ permutations of these p elements are indistinguishable.

☞ For practice: compute the number of hands of 5 cards drawn from a 32-card deck.

Properties for the binomial coefficient

The binomial coefficient $\binom{n}{p}$ satisfies the following properties:

- Symmetry: $\forall p \in \{0, 1, \ldots, n\}$, $\binom{n}{p} = \binom{n}{n-p}$.
- Pascal's Triangle: $\binom{n}{p} + \binom{n}{p+1} = \binom{n+1}{p+1}$
- Newton's binomial formula:
  $\forall n \in \mathbb{N}, \forall a, b \geq 0$, $(a+b)^n = \sum_{p=0}^{n} \binom{n}{p} a^p b^{n-p}$.