# TWO DISTANCE-BASED FAMILIES OF STATISTICAL TESTS OF UNIFORMITY FOR PROBABILITY MEASURES ON HOMOGENEOUS COMPACT POLISH SPACES

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We consider a compact Polish space $(\mathscr{X}, \mathrm{d})$ that is homogeneous in the sense that it can be equipped with a uniform probability measure $\mu_0$: a measure that puts the same mass to all balls sharing the same radius, for every radius (small enough). Given a sample of points on $\mathscr{X}$, we introduce and study two different distance-based families of statistical tests. The first ones test that the sample is uniformly spread on $\mathscr{X}$, at least as well as an i.i.d. sample from $\mu_0$, the second ones test that the sample is independently identically distributed (i.i.d.) from $\mu_0$. The tests statistics are based on distance-to-measure signatures, as introduced in Brécheteau (2019), and thus, on nearest neighbours computations. We investigate stability and discriminative properties of the signatures, provide separation rates for the tests and illustrate their performance on several examples including Riemanian manifolds involved in directional statistics or hyperbolic spaces. We compare the discordant behaviours of the two families of tests on non i.i.d. data sets, provide data-driven methods to select the best number of nearest neighbours, and propose an application to shape analysis.

## 1. Introduction.

1.1. *Testing uniformity and/or independence of data points on non Euclidean data.* Classical statistical methods are mostly designed to deal with Euclidean data, lying in $\mathbb{R}^d$. However, in many areas of applications, data lie in more general spaces like graphs or more general Riemannian manifolds such as the flat torus $\mathbb{T}^2$, hyperbolic spaces, or spaces dedicated to *directional data*: the circle $\mathbb{S}^1$, the 2-dimensional sphere $\mathbb{S}^2$, hyperspheres, rotational groups such as the Grassmannian $\mathbb{G}_{p,d}$ (the group of projection matrices on $p$-dimensional subspaces of $\mathbb{R}^d$) or the Stiefel manifold. A recent overview on advanced methods for directional data is available in Pewsey and García-Portugués (2021). This overview updates the overview of Jupp and Mardia (1989). Applications of statistical method on non Euclidean data in aeronautics for instance are available in Dai and Müller (2018); Le Brigant and Puechmorel (2019). More generaly, in Pewsey and García-Portugués (2021), the authors provide recent references of applications of statistical methods on directional data in the fields of bioinformatics, astronomy, medicine, genetics, neurology, image analysis, text mining, machine learning, the modelling of wildfires and sea conditions. Examples of data embedded into negatively curved spaces are available in Cabanes (2022) for instance, for radar and audio data, or in Klimovskaia et al. (2020), in biology, for single cells data.

The question of testing if a sample of points is uniformly distributed on a space originates from Bernoulli (1735). It is still an ongoing question. For data on compact Riemanian manifolds, both Sobolev tests Giné M. (1975) and tests based on nearest neighbours computations Ebner, Henze and Yukich (2018) have been investigated. For directional data, the question has been extensively tackled in the past decades. Testing uniformity in the context of directional data boils done testing that there is no priviledged direction in the data. A recent overview in this context is available in García-Portugués and Verdebout (2018), especially for the circle, the sphere, and the hypersphere. Sobolev tests, including the tests of

Beran (1968, 1969) (on homogenous spaces, and more specifically on the sphere), and of Giné M. (1975) (on general Riemannian manifolds) consists in projecting the data on the harmonics. Such tests encompass the tests of Rayleigh (1919) and of Bingham (1974). Both of these two tests were modified by Jupp (2001) to improve their convergence under the null hypothesis. Other tests based on the mean value of pairwise distances were developped in Pycke (2010); Bakshaev (2010). Refinements of Sobolev tests on $\mathbb{S}^1$ were designed in Bogdan, Bogdan and Futschik (2002) with a data-driven selection of the number of harmonics for the Sobolev tests. This method was generalised in the context of compact Riemannian manifolds in Jupp (2008, 2009). Uniformity tests in the high dimensional context have also been investigated in Paindaveine and Verdebout (2016); Cutting, Paindaveine and Verdebout (2017), or in the noisy context in Lacour and Pham Ngoc (2014) for $\mathbb{S}^2$. We refer the reader to García-Portugués and Verdebout (2018) for the many other existing tests for uniformity for directional data. The specific case of data in the Grassmannian have been tackled in Chikuse and Watson (1995); Chikuse (2003). Uniformity tests are particular instances of goodness-of-fit tests, such as tests based on kernels Gretton et al. (2012) or on optimal transport Hallin, Mordant and Segers (2021), that could also be used to test uniformity of data. A last remark is that there is no onnibus test that can be most powerful under any kind of alternatives Escanciano (2009). For instance, test procedures performant against unimodal alternatives are not necessarily performant against multimodal alternatives. Therefore, it could be interresting to consider several tests at the same time, and use an adaptative procedure to aggregate the tests, Fromont and Laurent (2006); Schrab et al. (2023).

Tests for uniformity on $[0,1]^d$ could be used to solve the problem of testing independence of coordinates, for i.i.d. continuous random vectors on $\mathbb{R}^d$, after pushing data through their marginal cumulative distribution functions. The question of testing the independence of the coordinates originates from Hoeffding (1948). Recent work based on copulas is available in Genest, Quessy and Remillard (2007). The question of testing serial dependence, that is, if successive data have been generated independently, has also been widely studied Hallin, Ingenbleek and Puri (1985), see also the runs tests in $\mathbb{R}^d$ of Marden (1999); Paindaveine (2009). However, the tough question of wether or not *a sample has been generated as a sample of $n$ points, independent, from the uniform distribution*, does not seem to have been considered in the litterature, as far as we are concerned. Such tests could be used to design and evaluate new procedures to generate i.i.d. uniform samples on general homogeneous spaces, e.g. Ratner (1987) in the context of dynamical systems. Although tests of uniformity mentioned in the previous paragraph could be used as bilateral tests to reject both non uniformity of samples and too important regularity of samples (like grids), most of these tests have not been designed for the purpose of testing independence, and power loss should be expected when considering non i.i.d alternatives. This is the problem we tackle in this paper, together with the problem of testing only homogeneity.

1.2. *Comparison of probability measures on homogeneous spaces.* Testing uniformity (and independence) on spaces demands a proper framework. In this paper, we will consider homogeneous spaces, on which a uniform probability measure exists. We require that these spaces to be compact and Polish, to allow the use of Wasserstein distances to metrize weak convergence. Before defining all of the notions and tools, we provide first notation that will be used throughout the paper.

Let $(\mathscr{X}, \mathrm{d})$ be a *compact Polish space*. For simplicity, we assume that $(\mathscr{X}, \mathrm{d})$ is a complete separable metric space. The *diameter* of $(\mathscr{X}, \mathrm{d})$ is defined as the maximal distance between two elements of $\mathscr{X}$: $\mathscr{D}(\mathscr{X}) = \max\{\mathrm{d}(x,y),\, x,y \in \mathscr{X}\}$. It is well-defined and finite since $(\mathscr{X}, \mathrm{d})$ is compact. The open ball centered at $x \in \mathscr{X}$ with radius $r > 0$ is denoted by $\mathrm{B}_{x,r} = \{y \in \mathscr{X},\, \mathrm{d}(x,y) < r\}$ and its closure by $\overline{\mathrm{B}}_{x,r} = \{y \in \mathscr{X},\, \mathrm{d}(x,y) \leq r\}$.

Let $\mathscr{P}(\mathscr{X})$ be the set of *Borel probability mesures* (also called *probability distributions*) on $(\mathscr{X},\mathrm{d})$. The *support* $\mathrm{Supp}(\mu)$ of a probability measure $\mu\in\mathscr{P}(\mathscr{X})$ is defined as the smallest closed subset of $\mathscr{X}$ with $\mu$-mass 1. A random variable $X$ defined on some probability space $(\Omega,\mathscr{F},P)$, with values on $(\mathscr{X},\mathrm{d})$, follows the distribution $\mu\in\mathscr{P}(\mathscr{X})$ if for all Borel set $B$ of $\mathscr{X}$, $P(X\in B)=\mu(B)$. The *expectation* of $f(X)$ for a $\mu$-integrable function $f$ is then defined by $\mathbb{E}_{X\sim\mu}[f(X)]=\mathbb{E}_\mu[f(X)]=\int_\Omega f(X(\omega))\mathrm{d}P(\omega)=\int_{\mathscr{X}}f(x)\mathrm{d}\mu(x)$. The *Dirac mass* at $x\in\mathscr{X}$, $\delta_x\in\mathscr{P}(\mathscr{X})$, is defined by $\delta_x(B)=\mathbb{1}_{x\in B}$, for every Borel set $B$ of $\mathscr{X}$, where $\mathbb{1}_{x\in B}$ is the indicator function equal to 1 if $x\in B$ and to 0 elsewise. For $n\in\mathbb{N}^*$, let $\mathscr{P}_n(\mathscr{X})=\{\frac{1}{n}\sum_{i=1}^n\delta_{x_i},\,x_1,\ldots,x_n\in\mathscr{X}\}$ be the set of uniform probability distributions supported on a set of $n$ points of $\mathscr{X}$. Both $\mathscr{P}(\mathscr{X})$ and $\mathscr{P}_n(\mathscr{X})$, equipped with the Wasserstein distance (c.f. Definition 1.2), are compact Polish spaces (Villani (2008), proof of Theorem 1.1 and Lemma S.5.3 in Brécheteau (2025)). Moreover, both $\mathcal{W}_1$ or $\mathcal{W}_2$ provide the same Borel $\sigma$-algebra on $\mathscr{P}(\mathscr{X})$ and $\mathscr{P}_n(\mathscr{X})$ since $\mathcal{W}_1\leq\mathcal{W}_2\leq\sqrt{\mathscr{D}(\mathscr{X})}\sqrt{\mathcal{W}_1}$. Therefore, we may define $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$, the space of Borel probability measures on $\mathscr{P}_n(\mathscr{X})$ equipped with the Wasserstein distance. Let $\boldsymbol{\mu}_n$ be a random measure with distribution $\mathbb{\mu}_n\in\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$. The expectation of $f(\boldsymbol{\mu}_n)$ is denoted by $\mathbb{E}_{\mathbb{\mu}_n}[f(\boldsymbol{\mu}_n)]=\mathbb{E}_{\boldsymbol{\mu}_n\sim\mathbb{\mu}_n}[f(\boldsymbol{\mu}_n)]$ and the probability of events depending on $\boldsymbol{\mu}_n$ by $\mathbb{P}_{\mathbb{\mu}_n}$ or $\mathbb{P}_{\boldsymbol{\mu}_n\sim\mathbb{\mu}_n}$. A random vector $(X_1,\ldots,X_n)$ is an *$n$-sample* from $\mu\in\mathscr{P}(\mathscr{X})$, when $X_1,\ldots,X_n$ are $n$ independent random variables with distribution $\mu\in\mathscr{P}(\mathscr{X})$. The distribution of an $n$-sample from $\mu$ is the product measure, denoted by $\mu^{\otimes n}$, and the random measure $\hat{\boldsymbol{\mu}}_n=\frac{1}{n}\sum_{i=1}^n\delta_{X_i}$ is the *empirical measure* associated to the $n$-sample $(X_1,\ldots,X_n)$. Its distribution is denoted by $\hat{\mathbb{\mu}}_n$. The expectation of $f(\hat{\boldsymbol{\mu}}_n)$ is denoted by $\mathbb{E}_{\hat{\mathbb{\mu}}_n}[f(\hat{\boldsymbol{\mu}}_n)]=\mathbb{E}_{\hat{\boldsymbol{\mu}}_n\sim\hat{\mathbb{\mu}}_n}[f(\hat{\boldsymbol{\mu}}_n)]$, or with a slight notational abuse, by $\mathbb{E}_\mu[f(\hat{\boldsymbol{\mu}}_n)]$, and the probability of events depending on $\hat{\boldsymbol{\mu}}_n$ by $\mathbb{P}_{\hat{\mathbb{\mu}}_n}$, $\mathbb{P}_{\hat{\boldsymbol{\mu}}_n\sim\hat{\mathbb{\mu}}_n}$ or $\mathbb{P}_\mu$.

Testing uniformity of samples of points makes sense on spaces where a uniform probability measure exists, that is, on homogeneous spaces. In the remaining of the paper, we will assume that the compact Polish space $(\mathscr{X},\mathrm{d})$ is $h_0$-homogeneous, as defined below.

DEFINITION 1.1 (Loomis (1945)). *A uniform probability measure $\mu_0$ on a Polish space $(\mathscr{X},\mathrm{d})$ is a Borel probability measure $\mu_0\in\mathscr{P}(\mathscr{X})$ that satisfies:*

$$\forall x,y\in\mathscr{X},\,\forall\epsilon>0,\,\mu_0(\mathrm{B}(x,\epsilon))=\mu_0(\mathrm{B}(y,\epsilon)),\tag{1.1}$$

*whereas for $h_0>l_0$ (with $l_0=\frac{1}{|\mathscr{X}|}$ if $\mathscr{X}$ is discrete with cardinality $|\mathscr{X}|$ and $l_0=0$ if $\mathscr{X}$ is not discrete), an $h_0$-uniform probability measure satisfies:*

$$\forall x,y\in\mathscr{X},\,\forall 0<r\leq\delta_{\mu_0,h_0},\,\mu_0(\mathrm{B}(x,r))=\mu_0(\mathrm{B}(y,r)),\tag{1.2}$$

*where $\delta_{\mu_0,h_0}=\delta_{\mu_0,h_0}(x)=\inf\{r>0,\,\mu_0(\overline{\mathrm{B}}(x,r))>h_0\}$ for some (and thus for all) $x\in\mathscr{X}$. It means that balls with the same radius, with mass at most $h_0$, have the same mass.*

*The space $(\mathscr{X},\mathrm{d})$ is ($h_0$-)homogeneous if an ($h_0$-)uniform probability measure $\mu_0$ exists.*

Such $h_0$-uniform probability measures do not always exist. For instance, for $h_0>l_0=\frac{1}{3}$, there is no $h_0$-uniform measure on the space $(\{a,b,c\},\mathrm{d})$ with $\mathrm{d}(a,b)=1$, $\mathrm{d}(a,c)=1$, $\mathrm{d}(b,c)=2$. There is no $h_0$-uniform measure on the segment $[0,1]\subset\mathbb{R}$, because of the boundary. However, a uniform measure exists on the circle and the sphere. An $h_0$-uniform measure also exists on the flat torus and the Bolza surface, for $h_0$ small enough with respect to their injectivity radius. These exemples are described in Section S.2.2 in Brécheteau (2025).

A second condition for the problem of testing uniformity to make sense is the unicity of such an $h_0$-uniform measure $\mu_0$. This unicity is given in Christensen (1970), for which we give a formulation in Theorem S.1.1 with a proof in Section S.5.4.1, both in Brécheteau (2025).

Given a random measure $\boldsymbol{\mu}_n$ on $\mathscr{P}_n(\mathscr{X})$, testing the homogeneity of its support requires a tool to compare $\boldsymbol{\mu}_n$ to the uniform measure $\mu_0$. In this paper, we will use the Wasserstein distances, distances between probability measures that metrize weak convergence.

DEFINITION 1.2 (Villani (2008)). *Let $(\mathscr{X}, \mathrm{d})$ be a compact Polish space. For $p \in [1, \infty)$, the $L_p$-Wasserstein distance $\mathcal{W}_{p,\mathrm{d}}$ (or $\mathcal{W}_p$) is defined for every $\mu, \nu \in \mathscr{P}(\mathscr{X})$ by:*

$$\mathcal{W}_{p,\mathrm{d}}^p(\mu,\nu) = \inf_{\pi \in \pi(\mu,\nu)} \int_{\mathscr{X}\times\mathscr{X}} \mathrm{d}^p(x,y)\mathrm{d}\pi(x,y), \tag{1.3}$$

*where $\pi(\mu,\nu)$ denotes the set of probability measures $\pi \in \mathscr{P}(\mathscr{X}\times\mathscr{X})$ with first marginal $\mu$ and second marginal $\nu$. The $L_p$-Wasserstein distance is the smallest value of $\mathbb{E}[\mathrm{d}(X,Y)^p]^{\frac{1}{p}}$ over all possible joint distributions $\pi \in \pi(\mu,\nu)$ of $(X,Y)$.*

When $\mathscr{X} = \mathbb{R}$, the Wasserstein distance $\mathcal{W}_p$ for $p \geq 1$ has an expression depending on the quantile functions $Q_\mu$ and $Q_\nu$ of $\mu$ and $\nu$ in $\mathscr{P}(\mathbb{R})$, given by (1.4). This formulation in terms of $L^p$ norm makes the Wasserstein distance convenient to compare distributions on $\mathbb{R}$.

$$\mathcal{W}_p^p(\mu,\nu) = \int_{u=0}^1 |Q_\mu(u) - Q_\nu(u)|^p \mathrm{d}u. \tag{1.4}$$

Almost sure weak convergence of the empirical measure to the sampling measure is established since the seminal works of Varadarajan (1958). Rates for $\mathbb{E}\left[\mathcal{W}_p(\hat{\boldsymbol{\mu}}_n,\mu)\right]$ or $\inf_{\mu_n \in \mathscr{P}_n(\mathscr{X})} \mathcal{W}_p(\mu_n,\mu)$ have been obtained in Weed and Bach (2019); Boissard and Le Gouic (2014); Bobkov and Ledoux (2019); Fournier and Guillin (2015); Mérigot, Santambrogio and Sarrazin (2021); Mérigot and Mirebeau (2016); Carlier, Delalande and Mérigot (2024); Le Gouic and Loubes (2017) in various contexts. From such bounds, we may deduce the convergence to zero in probability, uniformly in the sampling distribution $\mu \in \mathscr{P}(\mathscr{X})$, of $\mathcal{W}_2(\hat{\boldsymbol{\mu}}_n,\mu)$, in Theorem 1.1. Such a result has been proved for $\mathcal{W}_p$, $p \geq 1$, in (Hallin, Mordant and Segers, 2021, Theorem 1, Corollary 1) for measures in Euclidean space $\mathbb{R}^d$ under uniform integrability of the $p$-th order moment. Theorem 1.1 is more general in the sense that we consider a general Polish space, but more restrictive in the sense that we assume that this Polish space is compact. Therefore, the uniform integrability assumption of second-order moments in Hallin, Mordant and Segers (2021) is not required here since automatically satisfied. Theorem 1.1 is a consequence of the upper bound for the expectation in Boissard and Le Gouic (2014). In Section S.5.1.1 in Brécheteau (2025), we provide an alternative proof.

THEOREM 1.1. *For every compact Polish space $(\mathscr{X}, \mathrm{d})$, we have both:*

$$\lim_{n\to\infty} \sup_{\mu\in\mathscr{P}(\mathscr{X})} \mathbb{E}_\mu\left[\mathcal{W}_2^2(\hat{\boldsymbol{\mu}}_n,\mu)\right] = 0, \tag{1.5}$$

$$\forall \epsilon > 0,\ \lim_{n\to\infty} \sup_{\mu\in\mathscr{P}(\mathscr{X})} \mathbb{P}_\mu\left(\mathcal{W}_2^2(\hat{\boldsymbol{\mu}}_n,\mu) > \epsilon\right) = 0. \tag{1.6}$$

Another process to compare measures on spaces, or more generally metric measure spaces Gromov (2007), is to push them to $\mathscr{P}(\mathbb{R})$, to an image called a *signature*. The measures will be compared by comparing their signatures. Such signatures have been introduced and studied for clustering or testing purposes Osada et al. (2002); Brécheteau (2019), and were proved stable with respect to the (Gromov-)Wasserstein distance, Mémoli (2011).

1.3. *Contributions and organisation of the paper.* In this paper, we tacke the question of testing that an observed empirical measure $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i} \in \mathscr{P}_n(\mathscr{X})$ based on $n$ observations $x_1, \ldots, x_n$ on a compact homogeneous Polish space $\mathscr{X}$ is uniform on $\mathscr{X}$. More precisely, considering that $\mu_n$ is a realisation of a random measure $\boldsymbol{\mu}_n \sim \mathbb{\mu}_n$, with $\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$, we aim at testing the hypothesis

$$\mathrm{H}_0 \text{“}\mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}\text{”},$$

against several alternatives, where $\hat{\mathbb{p}}_{0,n}$ is the distribution of empirical measures $\hat{\boldsymbol{\mu}}_{0,n}$ based on i.i.d. $n$-samples from the uniform distribution $\mu_0$ on $\mathscr{X}$. In this paper, we will introduce two new families of tests, depending on some regularity parameter $h \in (0,1]$, corresponding to a proportion of nearest neighbours. The first family of tests, $(\phi^{\mathrm{hom}}_{n,h})_{h\in(0,1]}$, rejects hypothesis $\mathrm{H}_0$ when the sample $x_1,\ldots,x_n$ is not uniformly spread on the space $\mathscr{X}$, meaning that $\mu_n$ is far from $\mu_0$. For instance, such tests would not reject $\mathrm{H}_0$ for regular grids. The second family of tests, $(\phi^{\mathrm{iid}}_{n,h})_{h\in(0,1]}$, rejects hypothesis $\mathrm{H}_0$ when the sample $x_1,\ldots,x_n$ does not behave as an i.i.d sample from the uniform distribution $\mu_0$. Such tests would always reject $\mathrm{H}_0$ for regular grids, since points on a grid are not independent.

For this purpose, we first introduce the notion of DTM-signature, as in Brécheteau (2019), based on the distance-to-measure (DTM) of Chazal, Cohen-Steiner and Mérigot (2011) that depends on the parameter $h \in (0,1]$. In Proposition 2.3, we prove that the family of DTM-signatures characterise the uniform distribution, that makes it a relevant tool to test uniformity. We also provide examples of lower bounds for the distance between a signature and the signature of the uniform distribution. In particular, in Proposition 2.5, we consider the case of a measure with a density bounded from below, with respect to the uniform measure. In Proposition 2.1 and 2.2 we provide upper bounds for the distance between signatures in terms of Wasserstein distance and derive parametric rates in expectation and thus in probability for the pseudo-distance induced by signatures. Then, in Definition 2.3, we introduce the notion of barycenter signature, as the average empirical signature for i.i.d. $n$-samples from a distribution $\mu \in \mathscr{P}(\mathscr{X})$. We also investigate its stability and discriminative properties.

The tests $(\phi^{\mathrm{iid}}_{n,h})_{h\in(0,1]}$ are based on a comparison of the empirical signature to the barycenter signature, whereas the tests $(\phi^{\mathrm{hom}}_{n,h})_{h\in(0,1]}$ are based on a comparison of the empirical signature to the signature of the uniform distribution. They are both defined in Section 3.1. In particular, we prove their uniform asymptotic convergence in Theorem 3.1. This performance in terms of power, is a consequence of the uniform convergence of Wasserstein distance between measures and empirical measures, uniformly on $\mathscr{P}(\mathscr{X})$, recalled in Theorem 1.1. We also provide parametric upper bounds on the separation rates for quite general alternatives in Theorem 3.2, and provide lower bounds in $\frac{1}{n}$ for the separation rates for i.i.d. samples in Theorem 3.3. Finally, we illustrate the performance of the two families of tests, discuss the selection of the parameter $h$ by investigating both aggregation of tests strategies and multiple testing strategies. We compare our tests to lots of classical tests of uniformity available in the litterature, on the circle $\mathbb{S}^1$, the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$ and the Grassmannian, and consider an illustration in the field of shape analysis.

The paper is organized as follows. We provide definitions and study stability and discriminative properties for the DTM-signatures and the barycenter signatures in Section 2. We also give examples of signatures computations for the circle $\mathbb{S}^1$ in Section 2.3. In Section 3 we define and study the two families of statistical tests of uniformity. In particular, consistency, and separation rates for the tests are discussed in Section 3.2. Extensive numerical investigations on the tests are available in Section 4. Additional results, numerical illustrations and proofs are available in the supplementary material, Brécheteau (2025).

## 2. Stable and discriminative signatures.

We introduce DTM and barycenter signatures, provide main discrimination and stability properties, and focus on the example of $\mathbb{S}^1$.

### 2.1. *A family of signatures, to characterize uniformity of measures.*

2.1.1. *Generalities.* The *distance-to-measure functions* (DTM) are generalisations of distance functions to the support of a measure. They depend on some smoothing parameter $h \in [0,1]$ that makes them robust to Wasserstein noise in data. They are defined in terms of measures of balls, or equivalently, in terms of Wasserstein metrics.

DEFINITION 2.1 (Chazal, Cohen-Steiner and Mérigot (2011); Buchet et al. (2016)). *For $h \in (0,1]$ and $\mu \in \mathscr{P}(\mathscr{X})$, the distance-to-measure $\mathrm{d}_{\mu,h} : \mathscr{X} \mapsto [0, \mathscr{D}(\mathscr{X})]$ with regularity parameter $h$ is defined for every $x \in \mathscr{X}$ by:*

$$\mathrm{d}^2_{\mu,h}(x) = \frac{1}{h}\int_{l=0}^{h} \delta^2_{\mu,l}(x)\mathrm{d}l, \tag{2.1}$$

*with $\delta_{\mu,l}(x)$, the radius of the ball centered at $x$ with $\mu$-mass $h$:*

$$\delta_{\mu,l}(x) = \inf\{r > 0,\, \mu(\overline{\mathrm{B}}_{x,r}) > l\}. \tag{2.2}$$

*Equivalently, if $\mathrm{Sub}_{\mu,h} = \{\nu \in \mathscr{P}(\mathscr{X}),\, h\nu \leq \mu\}$ denotes the set of probability measures $\nu \in \mathscr{P}(\mathscr{X})$ so that $h\nu$ is a submeasure of $\mu$, and $\mu_{x,h} \in \mathrm{Sub}_{\mu,h}$ is any restriction of $\mu$ to the ball centered at $x$ with $\mu$-mass $h$, (more precisely, $h\mu_{x,h}$ coincides with $\mu$ on $\mathrm{B}_{x,\delta_{\mu,h}(x)}$ and is unique if the $\mu$-mass of the boundary of $\mathrm{B}_{x,\delta_{\mu,h}(x)}$ is null),*

$$\mathrm{d}_{\mu,h}(x) = \inf_{\nu \in \mathrm{Sub}_{\mu,h}} \mathcal{W}_2(\delta_x, \nu) = \mathcal{W}_2(\delta_x, \mu_{x,h}). \tag{2.3}$$

For $\mu_n \in \mathscr{P}_n(\mathscr{X})$, the distance to the measure $\mu_n$ at $x \in \mathscr{X}$ coincides with the mean squared distance from $x$ to its $q = nh$ first nearest neighbours $(x_j(x))_{1\leq j\leq q}$ in $\mathrm{Supp}(\mu_n)$:

$$\forall x \in \mathscr{X},\, \mathrm{d}^2_{\mu_n,h}(x) = \frac{1}{q}\sum_{j=1}^{q} \mathrm{d}^2(x, x_j(x)). \tag{2.4}$$

For discrete homogeneous spaces, the sequence of distances to nearest neighbours in $\mathscr{X}$, $d_1 = \mathrm{d}(x, x_1(x)) \leq d_2 = \mathrm{d}(x, x_2(x)) \leq \cdots \leq d_{|\mathscr{X}|} = \mathrm{d}(x, x_{|\mathscr{X}|}(x))$ does not depend on the point $x \in \mathscr{X}$. This property is still satisfied for continuous homogeneous spaces $\mathscr{X}$ in the sense that the family $(\delta_{\mu_0,h}(x))_{0\leq h\leq 1}$ does not depend on $x \in \mathscr{X}$. As a consequence, the distance to the measure $\mu_0$, $\mathrm{d}_{\mu_0,h}$, is constant. The same phenomenon appends for $h_0$-homogeneous spaces, until the parameter $h = h_0$:

$$\forall h \in [0, h_0], \exists \mathrm{d}_h \geq 0,\, \forall x \in \mathscr{X},\, \mathrm{d}_{\mu_0,h}(x) = \mathrm{d}_h. \tag{2.5}$$

For discrete spaces $\mathscr{X}$ with cardinality $N \in \mathbb{N}^*$ and $q = Nh$ an integer, $\mathrm{d}_h = \sqrt{\frac{1}{q}\sum_{i=1}^{q} d_i^2}$. Exact computation of $\mathrm{d}_h$ is given in Section 2.3 for the unit circle $\mathbb{S}^1$, and in Section S.2.2 in Brécheteau (2025) for the sphere $\mathbb{S}^2$, the flat torus $\mathbb{T}^2$, and the Bolza surface $\mathbb{B}$. DTM functions characterize the supports since $\mathrm{d}_{\mu,0}(x) = \delta_{\mu,0}(x) = 0$ for $x \in \mathrm{Supp}(\mu)$, satisfy stability properties Chazal, Cohen-Steiner and Mérigot (2011); Buchet et al. (2016):

$$\forall \mu, \nu \in \mathscr{P}(\mathscr{X}),\, \|\mathrm{d}_{\mu,h} - \mathrm{d}_{\nu,h}\|_\infty := \sup_{x\in\mathscr{X}} |\mathrm{d}_{\mu,h}(x) - \mathrm{d}_{\nu,h}(x)| \leq \frac{1}{\sqrt{h}}\mathcal{W}_2(\mu,\nu), \tag{2.6}$$

and are 1-Lipschitz, Chazal, Cohen-Steiner and Mérigot (2011); Buchet et al. (2016):

$$\forall \mu \in \mathscr{P}(\mathscr{X}),\, \forall x, y \in \mathscr{X},\, |\mathrm{d}_{\mu,h}(x) - \mathrm{d}_{\mu,h}(y)| \leq \mathrm{d}(x,y). \tag{2.7}$$

The $L_1$-*distance-to-measure signatures* have been first introduced in Brécheteau (2019), is the context of two-samples testing, with approximations based on a subsample of the data. In this paper, we define $L_2$-distance-to-measure signatures (DTM-signatures) with approximations based on the whole sample, and prove stability and discriminative results.

DEFINITION 2.2. *For $h \in [0,1)$ and $\mu \in \mathscr{P}(\mathscr{X})$, the distance-to-measure signature of $\mu$ with parameter $h$ is defined as the distribution of $\mathrm{d}_{\mu,h}(X)$, where $X$ is a random variable from $\mu$, that is, as the pushforward (denoted by $\#$) of $\mu$ by the $L_2$-DTM function $\mathrm{d}_{\mu,h}$:*

$$s_h(\mu) = \mathrm{d}_{\mu,h\#\mu} \in \mathscr{P}([0, \mathscr{D}(\mathscr{X})]). \tag{2.8}$$

*Thus, the DTM-signature to the measure* $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ *with parameter* $h$*, is defined by:*

$$s_h(\mu_n) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\mathrm{d}_{\mu_n,h}(x_i)} \in \mathscr{P}_n([0,\mathscr{D}(\mathscr{X})]). \tag{2.9}$$

2.1.2. *Stability results.* Signatures inherit the stability properties of the DTM.

PROPOSITION 2.1. *For every* $\mu$, $\nu \in \mathscr{P}(\mathscr{X})$*, we have both:*

$$\mathcal{W}_1(s_h(\mu), s_h(\mu_0)) \leq \mathcal{W}_2(s_h(\mu), s_h(\mu_0)) \leq \frac{1}{\sqrt{h}}\mathcal{W}_2(\mu,\mu_0), \tag{2.10}$$

$$\mathcal{W}_1(s_h(\mu), s_h(\nu)) \leq \mathcal{W}_2(s_h(\mu), s_h(\nu)) \leq \left(1+\frac{1}{\sqrt{h}}\right)\mathcal{W}_2(\mu,\nu). \tag{2.11}$$

A proof of Proposition 2.1 is available in Section S.5.2.1 in Brécheteau (2025).

The empirical distribution is strongly consistent in the Wasserstein distance in the sense that for every $\mu \in \mathscr{P}(\mathscr{X})$, $\mathcal{W}_2(\hat{\boldsymbol{\mu}}_n, \mu) \to 0$ almost surely as $n \to \infty$. This is a consequence of the almost surely weak convergence of the empirical measure $\hat{\boldsymbol{\mu}}_n$ to the sampling measure $\mu$ on the Polish space $\mathscr{X}$ Varadarajan (1958), and the fact that $\mathcal{W}_2$ metrizes weak convergence in compact Polish spaces (Villani, 2008, Theorem 6.9). It follows from Proposition 2.1 that:

$$\forall \mu \in \mathscr{P}(\mathscr{X}),\ \mathbb{P}_\mu\left(\lim_{n\to\infty} \mathcal{W}_1(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu)) = 0\right) = 1. \tag{2.12}$$

A direct consequence of Theorem 1.1 and Proposition 2.1 is the convergence to zero in probability, uniformly in the underlying distribution $\mu \in \mathscr{P}(\mathscr{X})$, of the $L_p$-Wasserstein distance between signatures and empirical signatures, for $p \in \{1,2\}$:

$$\forall \epsilon > 0,\ \lim_{n\to\infty} \sup_{\mu\in\mathscr{P}(\mathscr{X})} \mathbb{P}_\mu\left(\mathcal{W}_p(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu)) > \epsilon\right) = 0. \tag{2.13}$$

Rates for $\mathbb{E}\left[\mathcal{W}_p(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu))\right]$ with $p \in \{1,2\}$ can be derived from the sharp rates for $\mathbb{E}[\mathcal{W}_p^p(\hat{\boldsymbol{\mu}}_n, \mu)]$ Weed and Bach (2019), see also Fournier and Guillin (2015); Boissard and Le Gouic (2014). These rates are necessarily higher than rates for $\inf_{\mu_n\in\mathscr{P}_n(\mathscr{X})} \mathcal{W}_p^p(\mu_n,\mu)$, Mérigot, Santambrogio and Sarrazin (2021); Mérigot and Mirebeau (2016). In Proposition 2.2, we provide sharper bounds based on some arguments in Chazal, Massart and Michel (2016), based on bounds for Wasserstein distances in $\mathbb{R}$, in Bobkov and Ledoux (2019).

PROPOSITION 2.2. *Let* $h \in (0,1]$, $l_h > 0$.

***General case:*** *There exists some* $n_h \in \mathbb{N}$ *and* $C > 0$ *that does only depend on the diameter of* $\mathscr{X}$ *(not on* $h$ *nor on* $\mu$*) so that for every* $n \geq n_h$*:*

$$\sup_{\mu\in\mathscr{P}(\mathscr{X}),\, \inf_{x\in\mathscr{X}} \mathrm{d}_{\mu,h}(x) \geq l_h} \mathbb{E}_\mu\left[\mathcal{W}_1(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu))\right] \leq \frac{C}{hl_h\sqrt{n}}, \tag{2.14}$$

$$\sup_{\mu\in\mathscr{P}(\mathscr{X}),\, \inf_{x\in\mathscr{X}} \mathrm{d}_{\mu,h}(x) \geq l_h} \mathbb{E}_\mu\left[\mathcal{W}_2(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu))\right] \leq \frac{C}{hl_h\sqrt{n}} + \frac{C}{n^{\frac{1}{4}}}. \tag{2.15}$$

***Particular case with finite moment:*** *Let* $c > 0$*. For* $\mu \in \mathscr{P}(\mathscr{X})$*, if* $s_h(\mu)$ *has a density* $f_{\mu,h}$ *with respect to the Lebesgue measure on* $\mathbb{R}$ *and a cumulative distribution function* $F_{\mu,h}$*, let*

$$J_{\mu,h} = \int_{x=-\infty}^{+\infty} \frac{F_{\mu,h}(x)\left(1-F_{\mu,h}(x)\right)}{f_{\mu,h}(x)}\mathrm{d}x. \tag{2.16}$$

*Under these asumptions, there exists some $n_h \in \mathbb{N}$ and $C > 0$ that does only depend on the diameter of $\mathscr{X}$ and on $c$ so that for every $n \geq n_h$:*

$$\sup_{\mu \in \mathscr{P}(\mathscr{X}),\, J_{\mu,h} < c,\, \inf_{x \in \mathscr{X}} \mathrm{d}_{\mu,h}(x) \geq l_h} \mathbb{E}_\mu \left[\mathcal{W}_2(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu))\right] \leq \frac{C}{h l_h \sqrt{n}}. \tag{2.17}$$

***Case of the uniform measure*** $\mu_0$***:*** *For the uniform measure $\mu_0$, we get that for some constant $C > 0$, for every $n \geq n_h$:*

$$\mathbb{E}_{\mu_0}\left[\mathcal{W}_1(s_h(\hat{\boldsymbol{\mu}}_{0,n}), s_h(\mu_0))\right] \leq \mathbb{E}_{\mu_0}\left[\mathcal{W}_2(s_h(\hat{\boldsymbol{\mu}}_{0,n}), s_h(\mu_0))\right] \leq \frac{C}{h \mathrm{d}_h \sqrt{n}}, \tag{2.18}$$

*where $\mathrm{d}_h$ is the constant value of the DTM,* (2.5)*.*

A proof of Proposition 2.2 is available in Section S.5.2.2 in Brécheteau (2025). For $h \in (0,1]$, the assumption $\inf_{x \in \mathscr{X}} \mathrm{d}_{\mu,h}(x) \geq l_h$ for some $l_h > 0$ is satisfied if and only if no point $x \in \mathscr{X}$ has a $\mu$-mass larger or equal to $h$. Indeed, $\mathrm{d}_{\mu,h}$ is continuous (2.7) on the compact set $\mathscr{X}$, so it attains its minimum, that is equal to 0 if and only if $\mu(\{x\}) \geq h$ for some $x \in \mathscr{X}$.

2.1.3. *Discrimination properties.* In this section, we first prove that $\mu_0$ is characterized by its DTM-signatures. Then, we derive lower-bounds for the distance between the signature $s_h(\mu)$ and the signature $s_h(\mu_0)$ of $\mu_0$, for several alternative measures $\mu \in \mathscr{P}(\mathscr{X})$. Additional lower bounds are available in Section S.2.1 in Brécheteau (2025).

PROPOSITION 2.3. *For every ($h_0$)-homogeneous compact Polish space $(\mathscr{X}, \mathrm{d})$ (with $h_0 \in (l_0, 1]$), the ($h_0$)-uniform measure $\mu_0$, that is unique according to Theorem S.1.1 in Brécheteau (2025), is determined by its DTM-signatures:*

$$\forall H \in (l_0, 1], \forall \mu \in \mathscr{P}(\mathscr{X}), (\forall h \in [0, H], s_h(\mu) = s_h(\mu_0)) \Leftrightarrow \mu = \mu_0, \tag{2.19}$$

*with $l_0 = \frac{1}{|\mathscr{X}|}$ if $\mathscr{X}$ is discrete with cardinality $|\mathscr{X}|$, and $l_0 = 0$ if not.*

A proof of Proposition 2.3 is available in Section S.5.2.3 in Brécheteau (2025).

However, the ($h_0$)-uniform measure $\mu_0$ is not determined by the signature $s_0(\mu)$ with parameter $h = 0$ since $s_0(\mu) = \delta_0 = s_0(\mu_0)$ if and only if $\mu$ is supported on $\mathscr{X}$. For a given alternative measure $\mu \in \mathscr{P}(\mathscr{X})$, we may be interested in the set of parameters $h \in (0, 1]$ for which the signature $s_h(\mu)$ differs from $s_h(\mu_0)$. Tests based on these parameters (c.f. Section 3.1) will be powerful. In general, the set of small parameters $h$ for which the signature $s_h(\mu)$ coincides with $s_h(\mu_0)$ is discrete:

PROPOSITION 2.4. *If $(\mathscr{X}, \mathrm{d})$ is not discrete. If $\mu \neq \mu_0$, then, there exists $H > 0$ so that $(0, H] \cap \{h \in [0,1],\, s_h(\mu) = s_h(\mu_0)\}$ is discrete.*

The proof of Proposition 2.4 is available in Section S.5.2.4 in Brécheteau (2025).

Measures with a support different to $\mathscr{X}$ have a signature different to $s_h(\mu_0)$. In particular, the case of measures with a constant density on a compact set is studied in Section S.2.1.1 in Brécheteau (2025), with a lower bound for the distance between signatures. In the following, we deal with measures with a positively lower bounded density with respect $\mu_0$. The lower bound we get will be used to lower bound the separation rates of our tests, in Section 3.2.3.

PROPOSITION 2.5. *Let $\mu_l = l\mu_1 + (1-l)\mu_0$ for $l \in (0,1)$ and $\mu_1 \in \mathscr{P}(\mathscr{X})$. Then,*

$$\mathcal{W}_1(s_h(\mu_l), s_h(\mu_0)) \geq (1-l)\mu_0(A_{l,h,\mu_1}) \left|\mathrm{d}_{\frac{h}{1-l}} - \mathrm{d}_h\right| \sim_{l \to 0} C_{h,\mu_1} l,$$

*where* $A_{l,h,\mu_1} = \left\{x \in \mathscr{X} \mid \operatorname{Supp}(\mu_1) \cap \mathrm{B}\left(x, r_{\frac{h}{1-l}}\right) = \emptyset\right\}$ *is the set of points* $x$ *for which the ball centered at* $x$ *with* $\mu_l$*-mass* $h$ *is included in the complementary set of* $\operatorname{Supp}(\mu_1)$*, that is, with radius* $r_{\frac{h}{1-l}}$*, the constant value of the function* $\delta_{\mu_0,\frac{h}{1-l}}$*, and where* $C_{h,\mu_1} = C_h\mu_0\left(\bigcup_{l>0} A_{l,h,\mu_1}\right)$ *is a non negative constant with* $C_h = 0$ *if and only if* $\mathscr{X}$ *is discrete and* $h < \frac{1}{|\mathscr{X}|}$*, and* $\mathrm{d}_h$ *is the constant value of the function* $\mathrm{d}_{\mu_0,h}$*, as defined in* (2.5)*.*

Notice that a sufficient condition for the constant $C_{h,\mu_1}$ to be nonzero in Proposition 2.5 is that $\mathscr{X}$ is not discrete (so that both $C_h > 0$ and $h \in (0,1) \mapsto r_h$ is continuous) and for some $\epsilon > 0$ and $x \in \mathscr{X}$, $\mathrm{B}(x, r_h + \epsilon) \subset \operatorname{Supp}(\mu_1)^c$. The proof of Proposition 2.5 is available in Section S.5.2.5 in Brécheteau (2025).

2.2. *Barycenters of signatures, to characterize uniformity and independence.*

2.2.1. *Generalities.* For any probability measure $\mathbb{s}$ on the space of probability measures $\mathscr{P}([0,\mathscr{D}(\mathscr{X})])$, $\bar{\mathbb{s}} = \arg\min_{s\in\mathscr{P}([0,\mathscr{D}(\mathscr{X})])} \mathbb{E}_{\boldsymbol{s}\sim\mathbb{s}}\left[\mathcal{W}_2^2(\boldsymbol{s}, s)\right]$ is the *Wasserstein barycenter of* $\mathbb{s}$. It exists and is unique. It is defined as the probability measure $\bar{\mathbb{s}}$ which quantile function corresponds to the pointwize mean of the quantile functions $Q_{\boldsymbol{s}}$ (i.e. the generalised inverse of cumulative distribution functions) of the random measures $\boldsymbol{s} \sim \mathbb{s}$. Its expression is given by Agueh and Carlier (2011); Le Gouic and Loubes (2017):

$$\bar{\mathbb{s}} = \left(x \in [0,1] \mapsto \int Q_s(x)\mathrm{d}\mathbb{s}(s)\right)_{\#\lambda_{[0,1]}}, \tag{2.20}$$

where $\lambda_{[0,1]}$ is the Lebesgue measure on $[0,1]$ and $\#$ denotes the pushforward measure.

DEFINITION 2.3. *The DTM-signature barycenter* $\bar{s}_h(\mathbb{\mu}_n)$ *to* $\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ *is defined as the Wasserstein barycenter of the distribution* $s_h(\mathbb{\mu}_n)$ *of* $s_h(\boldsymbol{\mu}_n)$*, where* $\boldsymbol{\mu}_n \sim \mathbb{\mu}_n$*:*

$$\bar{s}_h(\mathbb{\mu}_n) = \arg\min_{s\in\mathscr{P}([0,\mathscr{D}(\mathscr{X})])} \mathbb{E}_{\boldsymbol{s}\sim s_h(\mathbb{\mu}_n)}\left[\mathcal{W}_2^2(s, \boldsymbol{s})\right]. \tag{2.21}$$

As above mentioned, $\bar{s}_h(\mathbb{\mu}_n)$ exists and is unique. When $\mathbb{\mu}_n = \hat{\mathbb{\mu}}_n$ is the distribution of empirical measures obtained from $n$-samples from a measure $\mu \in \mathscr{P}(\mathscr{X})$, the distribution $s_h(\hat{\mathbb{\mu}}_n)$ can be seen as the pushforward by $(x_1,\ldots,x_n) \mapsto s_h\left(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right)$ of the measure $\mu^{\otimes n}$ of an $n$-sample from $\mu$. In practice, empirical DTM-signatures barycenters are approximated by a Monte-Carlo procedure, c.f. Section S.4.1 in Brécheteau (2025).

2.2.2. *Stability results.* Barycenter signatures also inherit the DTM's stability properties.

PROPOSITION 2.6. *If* $\mu, \nu \in \mathscr{P}(\mathscr{X})$*, then,*

$$\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{\mu}}_n), \bar{s}_h(\hat{\mathbb{\nu}}_n)) \le \left(1 + \frac{1}{\sqrt{h}}\right)\mathcal{W}_2(\mu, \nu). \tag{2.22}$$

*Moreover, in general, if* $\mathbb{\mu}_n, \mathbb{\nu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$*, then,*

$$\mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_n), \bar{s}_h(\mathbb{\nu}_n)) \le \left(1 + \frac{1}{\sqrt{h}}\right)\mathcal{W}_1(\mathbb{\mu}_n, \mathbb{\nu}_n), \tag{2.23}$$

*where* $\mathcal{W}_1$ *is computed with respect to the* $L_2$*-Wasserstein distance on* $\mathscr{P}(\mathscr{X})$*.*

The proof of Proposition 2.6 is available in Section S.5.2.6 in Brécheteau (2025).

Empirical DTM-signature barycenters based on $n$-samples from a distribution $\mu \in \mathscr{P}(\mathscr{X})$ converge in distribution to the DTM-signature associated to $\mu$, when the sample size $n$ goes to $\infty$. This convergence is uniform on $\mu \in \mathscr{P}(\mathscr{X})$.

PROPOSITION 2.7. *The barycenter $\bar{s}_h(\hat{\mathbb{\mu}}_n)$ converges to $s_h(\mu)$, uniformly on $\mu \in \mathscr{P}(\mathscr{X})$, in the sense that for $p \in \{1,2\}$ and $h \in (0,1]$:*

$$\sup_{\mu\in\mathscr{P}(\mathscr{X})} \mathcal{W}_p(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n)) \to 0,\ n \to \infty. \tag{2.24}$$

*Moreover, for $h \in (0,1]$ and $l_h > 0$.*

***General case:*** *There exists some $n_h \in \mathbb{N}$ and $C > 0$ that does only depend on the diameter of $\mathscr{X}$ (not on $h$ nor on $\mu$) so that for every $n \geq n_h$:*

$$\sup_{\mu\in\mathscr{P}(\mathscr{X}),\, \inf_{x\in\mathscr{X}} \mathrm{d}_{\mu,h}(x)\geq l_h} \mathcal{W}_p(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n)) \leq \frac{C}{hl_h\sqrt{n}} + \frac{C}{n^{\frac{1}{4}}}. \tag{2.25}$$

***Particular case with finite moment:*** *Let $c > 0$. For $\mu \in \mathscr{P}(\mathscr{X})$, let $J_{\mu,h}$ be as in Proposition 2.2. Then, there exists some $n_h \in \mathbb{N}$ and $C > 0$ that does only depend on the diameter of $\mathscr{X}$ and on $c$ so that for every $n \geq n_h$:*

$$\sup_{\mu\in\mathscr{P}(\mathscr{X}),\, J_{\mu,h}<c,\, \inf_{x\in\mathscr{X}} \mathrm{d}_{\mu,h}(x)\geq l_h} \mathcal{W}_p(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n)) \leq \frac{C}{hl_h\sqrt{n}}. \tag{2.26}$$

***Case of the uniform measure $\mu_0$:*** *For the uniform measure $\mu_0$, we get that for some constant $C > 0$, for every $n \geq n_h$:*

$$\mathcal{W}_p(s_h(\mu_0), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})) \leq \frac{C}{h\mathrm{d}_h\sqrt{n}}. \tag{2.27}$$

*These results still hold when we replace $\mathcal{W}_p(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n))$ by $\mathbb{E}_\mu\left[\mathcal{W}_p(s_h(\hat{\boldsymbol{\mu}}_n), \bar{s}_h(\hat{\mathbb{\mu}}_n))\right]$ and when we replace $\mathcal{W}_p(s_h(\mu_0), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n}))$ by $\mathbb{E}_{\mu_0}\left[\mathcal{W}_p(s_h(\hat{\boldsymbol{\mu}}_{0,n}), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n}))\right]$.*

The proof of Proposition 2.7 is available in Section S.5.2.7 in Brécheteau (2025).

2.2.3. *Discrimination properties.* If the uniform measure is characterized by its DTM-signatures, according to Proposition 2.3, it follows from convergence of barycenter signatures in Proposition 2.7 that it is also characterized by its barycenter signatures, for small enough regularity parameters $h$ and large enough sample sizes $n$:

COROLLARY 2.1. *For every ($h_0$)-homogeneous (for $h_0 \in (l_0,1]$ with $l_0$ as in Proposition 2.3) compact Polish space $(\mathscr{X}, \mathrm{d})$, the ($h_0$)-uniform measure $\mu_0$ is determined by its barycenter signatures, in the sense that, for $s_0 = s_h(\mu_0)$ or $s_0 = \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$, we have that:*

$$\forall H \in (l_0,1], \forall \mu \in \mathscr{P}(\mathscr{X}), \left(\forall h \in [0,H],\ \lim_{n\to\infty} \mathcal{W}_2(\bar{s}_h(\hat{\mathbb{\mu}}_n), s_0) = 0\right) \Leftrightarrow \mu = \mu_0. \tag{2.28}$$

When $h \in (0,1]$ and $n \in \mathbb{N}$ are fixed, the question of whether for $\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ (resp. $\mu \in \mathscr{P}(\mathscr{X})$) the equality of the barycenter signature $\bar{s}_h(\mathbb{\mu}_n)$ (resp. $\bar{s}_h(\hat{\mathbb{\mu}}_n)$) with $\bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$ implies that $\mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}$ (resp. $\mu = \mu_0$) is more tricky. For instance, this is false for $h \leq \frac{1}{n}$ since in this case, $\bar{s}_h(\mathbb{\mu}_n) = \delta_0$ for any $\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$. Another example of measures in $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ with the same barycenter signature are the measure $\hat{\mathbb{\nu}}_n$ for $\nu = \delta_x$ for some $x \in \mathscr{X}$ and $\mathbb{\nu}_n$ supported on $\{\delta_x,\, x \in \mathscr{X}\}$. Indeed, $\bar{s}_h(\hat{\mathbb{\nu}}_n) = \delta_0 = \bar{s}_h(\mathbb{\nu}_n)$ but $\hat{\mathbb{\nu}}_n$ may be different to $\mathbb{\nu}_n$. Another example of two measures with the same barycenter signature, based on sample points regularly spaced on geodesics, is given by Example S.2.1 in Brécheteau (2025).

These examples enhance that the non uniformity of a measure is not necessarily detected from a barycenter signature with fixed parameters $h \in (0,1]$ and $n \in \mathbb{N}^*$. However, according to the stability of the signature barycenter in Proposition 2.6, for fixed $h \in (0,1]$

and $n \in \mathbb{N}$, any probability measure $\mu \in \mathscr{P}(\mathscr{X})$ close enough (in terms of the Wasserstein distance) to a probability measure $\nu$ so that $\bar{s}_h(\hat{\nu}_n) \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$ also satisfies that $\bar{s}_h(\hat{\mathbb{\mu}}_n) \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$. A simple example is the case of a measure $\nu = \delta_x$ for some $x \in \mathscr{X}$, for which $\bar{s}_h(\hat{\nu}_n) = \delta_0 \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$. As well, any probability measure $\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ close enough to a probability measure $\mathbb{\nu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ so that $\bar{s}_h(\mathbb{\nu}_n) \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$ also satisfies that $\bar{s}_h(\hat{\mathbb{\mu}}_n) \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$. A simple example is the case of a measure $\mathbb{\nu}_n$ supported on $\{\delta_x,\, x \in \mathscr{X}\}$, for which again, $\bar{s}_h(\mathbb{\nu}_n) = \delta_0 \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$. Moreover, on sets of measures $\mathscr{P}_{(\delta)}(\mathscr{X}) = \{\mu \in \mathscr{P}(\mathscr{X}),\, \mathcal{W}_2(s_h(\mu), s_h(\mu_0)) > \delta\}$ for some $\delta > 0$, detection of the uniform measure can be done, for a fixed parameter $h \in (0,1]$ and for every sample size larger than some $N \in \mathbb{N}^*$ that does not depend on the measure in $\mathscr{P}_{(\delta)}$:

$$\forall h \in (0,1],\, \exists N \in \mathbb{N}^*,\, \forall n \geq N,\, \forall \mu \in \mathscr{P}_{(\delta)}(\mathscr{X}),\, \bar{s}_h(\hat{\mathbb{\mu}}_n) \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n}). \tag{2.29}$$

This follows directly from the uniform convergence over $\mathscr{P}(\mathscr{X})$ of the barycenter signature to the sampling signature, given by Proposition 2.7. This result applies to alternatives with support different to $\mathscr{X}$ detailed in Section S.2.1 in Brécheteau (2025), as well as with measures with positively lower-bounded density with respect to $\mu_0$, in Proposition 2.5.

Finally, we provide a non trivial computation of the distance between two barycenter signatures, for $\hat{\mathbb{\mu}}_{0,n}$ and a mixture involving $\hat{\mathbb{\mu}}_{0,n}$.

PROPOSITION 2.8. *Let $h \in [0,1]$, let $\mathbb{\mu}_{l,n} = l\mathbb{\mu}_{1,n} + (1-l)\hat{\mathbb{\mu}}_{0,n}$ be a mixture of two probability distributions on $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$: some probability distribution $\mathbb{\mu}_{1,n} \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ with probability $l \in (0,1)$ and the distribution of an i.i.d. $n$-sample from $\mu_0$ with probability $1-l$. Then,*

$$\mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_{l,n}), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})) = l\mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_{1,n}), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})). \tag{2.30}$$

In Proposition 2.8, the distance between the two barycenters $\mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_{1,n}), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n}))$ is nonzero for instance when $\mathbb{\mu}_{1,n}$ is a Dirac mass on $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ for fixed $x_1, \ldots, x_n \in \mathscr{X}$ satisfying $s_h\left(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right) \neq \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$. This occurs when $h > \frac{1}{n}$, for the unit circle $\mathscr{X} = \mathbb{S}^1$ and $x_1, \ldots, x_n$ a regular grid on $\mathscr{X}$. Indeed, in this case, $s_h\left(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right)$ is a Dirac mass, unlike $\bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$. The proof of Proposition 2.8 is available in Section S.5.2.8 in Brécheteau (2025).

2.3. *Computation of signatures on the unit circle* $\mathbb{S}^1$. In this section, we compute the sampling signatures $s_h(\mu_0)$ for the uniform measure $\mu_0$ on $\mathbb{S}^1$. We plot the quantile functions of these signatures, barycenter signatures $\bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$ as well as several empirical signatures $s_h(\hat{\boldsymbol{\mu}}_{0,n})$ based on $n$-samples from $\mu_0$, in Figure 1, with parameters $n \in \{20, 50, 100, 1000\}$ and $h \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. The sampling (true) signature $s_h(\mu_0)$ is a Dirac mass $\delta_{\mathrm{d}_h}$. Its quantile function is a line with ordinate $\mathrm{d}_h = \pi h/\sqrt{3}$, see Figure 1 (left).

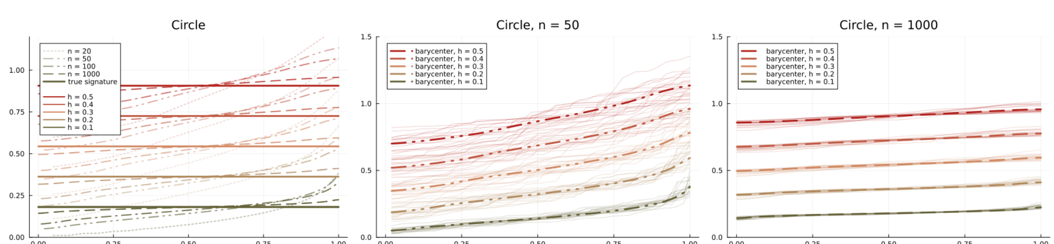


FIG 1. *Signatures and barycenter signatures for the unit circle* $\mathbb{S}^1$

We observe the convergence of the barycenter signatures $\bar{s}_h(\hat{\mathbb{µ}}_{0,n})$ to $s_h(\mu_0)$ when the sample size goes to infinity in Figure 1 (left). We also observe a reduction of the variance of the empirical signatures $s_h(\hat{\boldsymbol{\mu}}_{0,n})$, when the sample size $n$ goes to infinity, in Figure 1 (middle, right).

The formula for $\mathrm{d}_h$ follows from the fact that for $x \in \mathbb{S}^1$ and $r \in [0,\pi]$, $\mu_0(\mathrm{B}_{x,r}) = r/\pi$. So, $\delta_{\mu_0,h}(x) = h\pi$ and $\mathrm{d}_h^2 = \mathrm{d}_{\mu_0,h}^2(x) = \pi^2 h^2/3$, according to (2.1). Additional computations of sampling signatures for the sphere $\mathbb{S}^2$, the flat torus $\mathbb{T}^2$ and the Bolza surface $\mathbb{B}$ are given in Section S.2.2 in Brécheteau (2025). A remark is that for 1-dimensional spaces such as $\mathbb{S}^1$, $\mathrm{d}_h$ is of order $h$, whereas for surfaces such as $\mathbb{S}^2$, $\mathbb{T}^2$ and $\mathbb{B}$, $\mathrm{d}_h$ is of order $\sqrt{h}$.

## 3. Statistical tests for uniformity.

3.1. *Definition of two families of statistical tests.* In this section, given a random distribution $\boldsymbol{\mu}_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ with distribution $\mathbb{µ}_n$ on $\mathscr{P}_n(\mathscr{X})$, we define two families of statistical tests $(\boldsymbol{\phi}_{n,h}^{\mathrm{hom}})_{0<h<1}$ and $(\boldsymbol{\phi}_{n,h}^{\mathrm{iid}})_{0<h<1}$, of the null hypothesis

$$\mathrm{H}_0 : \mathbb{µ}_n = \hat{\mathbb{µ}}_{0,n},$$

when $X_1, \ldots, X_n$ is an i.i.d. $n$-sample from $\mu_0$, against the alternative hypothesis

$$\mathrm{H}_1 : \mathbb{µ}_n \neq \hat{\mathbb{µ}}_{0,n}.$$

In the sequel we may consider alternatives based on distance to the true signature $s_h(\mu_0)$, to the barycenter signature $\bar{s}_h(\hat{\mathbb{µ}}_{0,n})$, or even alternatives based on *median signatures*. A median signature $\bar{s}_{h,1}(\mathbb{µ}_n)$ of $\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ is defined as a Wasserstein median (Carlier, Chenchene and Eichinger, 2023, Section 3.1) of the distribution $s_h(\boldsymbol{\mu}_n)$ for $\boldsymbol{\mu}_n \sim \mathbb{µ}_n$:

$$\bar{s}_{h,1}(\mathbb{µ}_n) \in \arg\min_{s \in \mathscr{P}([0,\mathscr{D}(\mathscr{X})])} \mathbb{E}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n}\left[\mathcal{W}_1(s, s_h(\boldsymbol{\mu}_n))\right]. \tag{3.1}$$

Additional details on median signatures are given in Section S.2.3 of Brécheteau (2025).

We may consider global alternatives on $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$. Let $(\mathrm{c}_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers, with $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, wet set:

$$\mathrm{H}_1(\hat{\mathbb{µ}}_{0,n}, \mathcal{W}_1, h, \epsilon_n, \mathrm{c}_n) = \{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})) \mid \mathcal{W}_1(\bar{s}_{h,1}(\mathbb{µ}_n), \bar{s}_{h,1}(\hat{\mathbb{µ}}_{0,n})) \geq \mathrm{c}_n,\ \mathcal{V}_{1,h}(\mathbb{µ}_n) \leq \epsilon_n\}, \tag{3.2}$$

$$\mathrm{H}_1(\hat{\mathbb{µ}}_{0,n}, \mathcal{W}_2, h, \epsilon_n, \mathrm{c}_n) = \{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})) \mid \mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), \bar{s}_h(\hat{\mathbb{µ}}_{0,n})) \geq \mathrm{c}_n,\ \mathcal{V}_{2,h}(\mathbb{µ}_n) \leq \epsilon_n\}, \tag{3.3}$$

for $\mathcal{V}_{1,h}(\mathbb{µ}_n) = \mathbb{E}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n}\left[\mathcal{W}_1(\bar{s}_{h,1}(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n))\right]$ and $\mathcal{V}_{2,h}(\mathbb{µ}_n) = \mathbb{E}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n}\left[\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n))\right]$.

We may also consider alternatives on $\mathscr{P}(\mathscr{X})$, when the observations are assumed to be i.i.d. samples. For $l_h > 0$, and a sequence of positive real numbers $(\mathrm{c}_n)_{n\in\mathbb{N}}$, we set:

$$\begin{aligned}\mathrm{H}_1(\mu_0, \mathcal{W}_1, l_h, \mathrm{c}_n) = \{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})) \mid \mathbb{µ}_n = \hat{\mathbb{µ}}_n \text{ for some } \mu \in \mathscr{P}(\mathscr{X}) \text{ s.t.}\\ \mathcal{W}_1(s_h(\mu_0), s_h(\mu)) \geq \mathrm{c}_n,\ \inf_{x \in \mathrm{Supp}(\mu)}\{\mathrm{d}_{\mu,h}(x)\} \geq l_h\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathrm{H}_1(\mu_0, \mathcal{W}_2, l_h, \mathrm{c}_n) = \{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})) \mid \mathbb{µ}_n = \hat{\mathbb{µ}}_n \text{ for some } \mu \in \mathscr{P}(\mathscr{X}) \text{ s.t.}\\ \mathcal{W}_2(s_h(\mu_0), s_h(\mu)) \geq \mathrm{c}_n,\ \inf_{x \in \mathrm{Supp}(\mu)}\{\mathrm{d}_{\mu,h}(x)\} \geq l_h\},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathrm{H}_1(\mu_0, \mathcal{W}_2, l_h, \mathrm{c}_J, \mathrm{c}_n) = \{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})) \mid \mathbb{µ}_n = \hat{\mathbb{µ}}_n \text{ for some } \mu \in \mathscr{P}(\mathscr{X}) \text{ s.t.}\\ \mathcal{W}_2(s_h(\mu_0), s_h(\mu)) \geq \mathrm{c}_n,\ \inf_{x \in \mathrm{Supp}(\mu)}\{\mathrm{d}_{\mu,h}(x)\} \geq l_h,\ J_{\mu,h} \leq \mathrm{c}_J\},\end{aligned} \tag{3.6}$$

where $J_{\mu,h}$ is defined in (2.16).

In Section 4.3, we will also investivate the discordant behaviours of the two families of statistical tests under homogeneous alternatives such as measures supported on regular grids. The tests $(\boldsymbol{\phi}_{n,h}^{\mathrm{hom}})_{0<h<1}$ will tend to never reject $\mathrm{H}_0$, whereas the tests $(\boldsymbol{\phi}_{n,h}^{\mathrm{iid}})_{0<h<1}$ will tend to always reject $\mathrm{H}_0$.

3.1.1. *A family of statistical tests of homogeneity of the sample.* Given a random probability distribution $\boldsymbol{\mu}_n \sim \mathbb{\mu}_n$ in $\mathscr{P}_n(\mathscr{X})$, we define the test statistic as:

$$\mathbf{T}_{n,h}^{\mathrm{hom}} = \mathcal{W}_1(s_h(\boldsymbol{\mu}_n), s_h(\mu_0)), \tag{3.7}$$

and the test as:

$$\boldsymbol{\phi}_{n,h}^{\mathrm{hom}} = \mathbb{1}_{\mathbf{T}_{n,h}^{\mathrm{hom}} > \mathrm{q}_{1-\alpha,n,h}^{\mathrm{hom}}}, \tag{3.8}$$

where $\mathrm{q}_{1-\alpha,n,h}^{\mathrm{hom}}$ is the $1-\alpha$-quantile of $\mathbf{T}_{n,h}^{\mathrm{hom}}$ when $\mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}$, that is, such that:

$$\mathrm{q}_{1-\alpha,n,h}^{\mathrm{hom}} = \inf\left\{c \in \mathbb{R},\, \mathbb{P}_{\mu_0}\left(\mathbf{T}_{n,h}^{\mathrm{hom}} \leq c\right) \geq 1-\alpha\right\}. \tag{3.9}$$

3.1.2. *A family of statistical tests of homogeneity and independence.* Given a random probability distribution $\boldsymbol{\mu}_n \sim \mathbb{\mu}_n$ in $\mathscr{P}_n(\mathscr{X})$, we define the test statistic as:

$$\mathbf{T}_{n,h}^{\mathrm{iid}} = \mathcal{W}_2(s_h(\boldsymbol{\mu}_n), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})), \tag{3.10}$$

where $\bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$ is the empirical DTM-signature barycenter, defined in Section 2.2, and the test as

$$\boldsymbol{\phi}_{n,h}^{\mathrm{iid}} = \mathbb{1}_{\mathbf{T}_{n,h}^{\mathrm{iid}} > \mathrm{q}_{1-\alpha,n,h}^{\mathrm{iid}}}, \tag{3.11}$$

where $\mathrm{q}_{1-\alpha,n,h}^{\mathrm{iid}}$ is the $1-\alpha$-quantile of $\mathbf{T}_{n,h}^{\mathrm{iid}}$ when $\mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}$, that is, such that:

$$\mathrm{q}_{1-\alpha,n,h}^{\mathrm{iid}} = \inf\left\{c \in \mathbb{R},\, \mathbb{P}_{\mu_0}\left(\mathbf{T}_{n,h}^{\mathrm{iid}} \leq c\right) \geq 1-\alpha\right\}. \tag{3.12}$$

3.2. *Consistency and separation rates for the tests.* A statistical test $\boldsymbol{\phi} = \phi(\boldsymbol{\mu}_n)$ is a $\{0,1\}$-valued random variable, that is a measurable function of the uniform measure on $n$ $\mathscr{X}$-valued observations, $\boldsymbol{\mu}_n \in \mathscr{P}_n(\mathscr{X})$. Based on these observations, the test $\boldsymbol{\phi}$ provides the decision either of rejecting $\mathrm{H}_0$ ($\boldsymbol{\phi} = 1$) or of not rejecting $\mathrm{H}_0$ ($\boldsymbol{\phi} = 0$). The performance of a statistical test $\boldsymbol{\phi}$ is measured in terms of its type I error:

$$\alpha_n(\boldsymbol{\phi}) = \mathbb{P}_{\hat{\mathbb{\mu}}_{0,n}}(\boldsymbol{\phi} = 1). \tag{3.13}$$

This type I error is usually fixed to be smaller than $\alpha \in (0,1)$ (for instance $\alpha = 5\%$). A test is of level $\alpha$ if its type I error is upper bounded by $\alpha$. The type II error, defined below, measures the ability of the test to detect the alternatives:

$$\beta_n(\boldsymbol{\phi}, params) = \sup_{\mathbb{\mu}_n \in \mathrm{H}_1(params)} \mathbb{P}_{\mathbb{\mu}_n}(\boldsymbol{\phi} = 0). \tag{3.14}$$

Following Ingster (1993); Faÿ et al. (2013) framework for the asymptotic theory of minimax tests, we define the *separation rate* as a sequence $(r_n)_{n\in\mathbb{N}}$ satisfying both, for $\mathrm{H}_1(c_n) = \{\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\, \mathbb{d}(\mathbb{\mu}_n, \hat{\mathbb{\mu}}_{0,n}) > c_n\}$ for some $c_n > 0$ and some pseudo-distance $\mathbb{d}$:

- For every sequence $(r'_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} r'_n/r_n = 0$:

  $$\lim_{n\to\infty} \inf_{\boldsymbol{\phi}} \left\{\alpha_n(\boldsymbol{\phi}) + \beta_n(\boldsymbol{\phi}, r'_n)\right\} = 1, \tag{3.15}$$

  where the infimum if taken over every $\{0,1\}$-valued measurable function of $\boldsymbol{\mu}_n$, $\boldsymbol{\phi}$.

- For every $\alpha, \beta > 0$, there exists $C > 0$ and a test $\boldsymbol{\phi}^*$ such that:

$$\limsup_{n\to\infty} \alpha_n(\boldsymbol{\phi}^*) \leq \alpha \text{ and } \limsup_{n\to\infty} \beta_n(\boldsymbol{\phi}^*, Cr_n) \leq \beta. \tag{3.16}$$

The first condition states that for rates faster than $r_n$, no test can perform better than a blind test for which the sum of the two risks is one. The second condition states that the test $\boldsymbol{\phi}^*$ is efficient for this separation rate in the sense that the sum of the two risks can be made arbitrarily small. The rate $r_n = n^{-1/2}$ is the usual rate in the regular parametric setting. In the sequel, we prove in Theorem 3.3 that no test can have a separation rate faster than $n^{-1}$ for hypothesis $\mathrm{H}_0$, against alternatives based on i.i.d. samples. In Theorem 3.2, we prove that our two families of tests $(\boldsymbol{\phi}_{n,h}^{\mathrm{hom}})_{h\in(0,1]}$ and $(\boldsymbol{\phi}_{n,h}^{\mathrm{iid}})_{h\in(0,1]}$ have a separation rate of at most $n^{-1/2}$ (or $n^{-1/4}$) under some general alternatives.

3.2.1. *Consistency.* For every $h \in (0,1]$, the tests $(\boldsymbol{\phi}_{n,h}^{\mathrm{hom}})_{n\in\mathbb{N}}$ and $(\boldsymbol{\phi}_{n,h}^{\mathrm{iid}})_{n\in\mathbb{N}}$ are asymptotically consistent, in the sense that the power converges uniformly to 1, under any alternative defined in Section 3.1 for a sequence $(\mathrm{c}_n)_{n\in\mathbb{N}}$ constant, equal to some value $\mathrm{c} > 0$.

THEOREM 3.1. *Let $h \in (0,1]$. Let $(\mathbf{T}_{n,h}, \mathrm{q})$ denote $(\mathbf{T}_{n,h}^{\mathrm{hom}}, \mathrm{q}_{1-\alpha,n,h}^{\mathrm{hom}})$ or $(\mathbf{T}_{n,h}^{\mathrm{iid}}, \mathrm{q}_{1-\alpha,n,h}^{\mathrm{iid}})$. For every $\mathrm{c} > 0$, and every sequence of positive real numbers $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0,*

$$\lim_{n\to\infty} \inf_{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\, \mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), \bar{s}_h(\hat{\mathbb{µ}}_{0,n})) > \mathrm{c},\, \mathcal{V}_{2,h}(\mathbb{µ}_n) \leq \epsilon_n} \mathbb{P}_{\mathbb{µ}_n}(\mathbf{T}_{n,h} > \mathrm{q}) = 1, \tag{3.17}$$

*where $\mathcal{V}_{2,h}(\mathbb{µ}_n)$ is defined in Section 3.1.*

*More generally, if $(\mathbb{µ}_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ so that $\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n))$ converges in probability to 0, and so that $\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), \bar{s}_h(\hat{\mathbb{µ}}_{0,n})) > \mathrm{c}$ for every $n \in \mathbb{N}$, then the power $(\mathbb{P}_{\mathbb{µ}_n}(\mathbf{T}_{n,h} > \mathrm{q}))_{n\in\mathbb{N}}$ converges to 1, when $n$ goes to $\infty$.*

*Consequently, for every $\mathrm{c} > 0$,*

$$\lim_{n\to\infty} \inf_{\mu\in\mathscr{P}(\mathscr{X}),\, \mathcal{W}_1(s_h(\mu), s_h(\mu_0)) > \mathrm{c}} \mathbb{P}_\mu(\mathbf{T}_{n,h} > \mathrm{q}) = 1. \tag{3.18}$$

*Moreover, for parameters $h \in \mathscr{H}(\mathscr{X})$, for every $\mathrm{c} > 0$,*

$$\lim_{n\to\infty} \inf_{\mu\in\mathscr{P}(\mathscr{X}),\, \mathcal{W}_2(\mu,\mu_0) > \mathrm{c}} \mathbb{P}_\mu(\mathbf{T}_{n,h} > \mathrm{q}) = 1, \tag{3.19}$$

*where $\mathscr{H}(\mathscr{X})$ is defined as the set of parameters $h$ that characterise the uniform distribution:*

$$\mathscr{H}(\mathscr{X}) := \{h \in [0,1],\, \forall \mu \in \mathscr{P}(\mathscr{X}),\, s_h(\mu) = s_h(\mu_0) \Rightarrow \mu = \mu_0\}. \tag{3.20}$$

The proof of Theorem 3.1 is available in Section S.5.3.1 in Brécheteau (2025).

The asymptotic consistency under the same kind of alternatives as in (3.19) has been proven in (Hallin, Mordant and Segers, 2021, Proposition 3) for their test based on the Wasserstein distance, on $\mathbb{R}^d$.

The consistency given by (3.18) is a direct consequence of the consistency given by (3.17). Indeed, the measures $(\hat{\mathbb{µ}}_n)_{n\in\mathbb{N}}$ defined from $\mu \in \mathscr{P}(\mathscr{X})$ satisfy the assumption that $\mathcal{V}_{2,h}(\hat{\mathbb{µ}}_n) \to 0, n \to \infty$, uniformly on $\mu \in \mathscr{P}(\mathscr{X})$, according to Proposition 2.1, Theorem 1.1, and to the convergence of barycenter signatures to the sampling signatures, uniformly on $\mu \in \mathscr{P}(\mathscr{X})$, according to Proposition 2.7. The assumption $\mathcal{V}_{2,h}(\mathbb{µ}_n) \to 0, n \to \infty$ is trivially satisfied for $\mathbb{µ}_n$ supported on a subset of measures in $\mathscr{P}_n(\mathscr{X})$ with the same signature, since then, $\mathcal{V}_{2,h}(\mathbb{µ}_n) = 0$. For instance, this occurs for $\mathbb{µ}_n = \delta_{\delta_x}$, $\mathbb{µ}_n = \alpha\delta_{\delta_x} + (1-\alpha)\delta_{\delta_y}$ for some $x, y \in \mathscr{X}$ and $\alpha > 0$, or $\mathbb{µ}_n = \alpha\delta_{\mu_n} + (1-\alpha)\delta_{\phi_{\#\mu_n}}$ for some isomorphism $\phi$ of $\mathscr{X}$ (i.e. so that $\mathrm{d}(\phi(x), \phi(y)) = \mathrm{d}(x,y)$ for every $x, y \in \mathscr{X}$) and some fixed measure $\mu_n \in \mathscr{P}_n(\mathscr{X})$. More generally, according to Definition 2.3 and Proposition 2.1, the assumption $\mathcal{V}_{2,h}(\mathbb{\mu}_n) \to 0, n \to \infty$ is satisfied for sequences of measures $(\mathbb{\mu}_n)_{n\in\mathbb{N}}$ with support with a Wasserstein diameter $\sup_{\mu_n,\nu_n\in\mathrm{Supp}(\mathbb{\mu}_n)} \mathcal{W}_2(\mu_n,\nu_n)$ converging to zero. However, the assumption $\mathcal{V}_{2,h}(\mathbb{\mu}_n) \to 0, n \to \infty$ is not always satisfied, and when this assumption is not satisfied, the power may not converge to 1, so that the convergence in (3.17) does not hold. For instance, for $\mathbb{\mu}_n = \frac{1}{2}\delta_{\mu_x} + \frac{1}{2}\hat{\mathbb{\mu}}_{0,n}$ for some $x \in \mathscr{X}$, we get that $\mathcal{V}_{2,h}(\mathbb{\mu}_n) \geq \frac{1}{2}\mathcal{W}_2(\delta_0, \bar{s}_h(\mathbb{\mu}_n)) = \frac{1}{4}\mathcal{W}_2(\delta_0, \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})) \to_{n\to\infty} \frac{\mathrm{d}_h}{4} > 0$, where we recall that $s_h(\mu_0) = \delta_{\mathrm{d}_h}$. Moreover, despite $\lim_{n\to\infty} \mathcal{W}_2(\bar{s}_h(\hat{\mathbb{\mu}}_{0,n}), \bar{s}_h(\mathbb{\mu}_n)) = \frac{\mathrm{d}_h}{2} > 0$ according to Proposition 2.8, we get that $\mathbb{P}_{\mathbb{\mu}_n}(\mathbf{T}_{n,h} > \mathrm{q}) \nrightarrow_{n\to\infty} 1$ since for every $n \in \mathbb{N}$, $\mathbb{P}_{\mathbb{\mu}_n}(\mathbf{T}_{n,h} > \mathrm{q}) \leq \mathbb{P}_{\boldsymbol{\mu}_n\sim\mathbb{\mu}_n}(\boldsymbol{\mu}_n = \delta_x) + \mathbb{P}_{\boldsymbol{\mu}_n\sim\mathbb{\mu}_n}(\boldsymbol{\mu}_n \neq \delta_x)\mathbb{P}_{\boldsymbol{\mu}_n\sim\mathbb{\mu}_n}(\mathbf{T}_{n,h} > \mathrm{q}|\boldsymbol{\mu}_n \neq \delta_x)$, and therefore, $\limsup_{n\to\infty} \mathbb{P}_{\mathbb{\mu}_n}(\mathbf{T}_{n,h} > \mathrm{q}) \leq \frac{1}{2} + \frac{\alpha}{2} = \frac{1+\alpha}{2} < 1$, for a continuous space $\mathscr{X}$ so that $\mathbb{P}_{\mu_0}(\mathbf{T}_{n,h} > \mathrm{q}) = \alpha$.

3.2.2. *An upper bound on the separation rate for general alternatives.* The $1-\alpha$-quantile $\mathrm{q}_{1-\alpha,n,h}$ ($\mathrm{q}^{\mathrm{hom}}_{1-\alpha,n,h}$ resp. $\mathrm{q}^{\mathrm{iid}}_{1-\alpha,n,h}$) of the test statistic $\mathbf{T}_{n,h}$ ($\mathbf{T}^{\mathrm{hom}}_{n,h}$ resp. $\mathbf{T}^{\mathrm{iid}}_{n,h}$) under the null hypothesis satisfies for some constant $C > 0$:

$$\forall h \in (0,1], \forall \alpha > 0, \exists n_h \in \mathbb{N}, \forall n \geq n_h, \mathrm{q}_{1-\alpha,n,h} \leq \frac{C}{h\mathrm{d}_h\alpha\sqrt{n}}, \tag{3.21}$$

where $\mathrm{d}_h$ is defined in (2.5). This is a consequence of Proposition 2.2, (2.18) for $\mathrm{q}^{\mathrm{hom}}_{1-\alpha,n,h}$, of Proposition 2.7, (2.27) for $\mathrm{q}^{\mathrm{iid}}_{1-\alpha,n,h}$, and of the Markov inequality. As a consequence, we prove in the following Theorem 3.2 that the statistical test is powerful at a parametric separation rate under mild assumptions:

THEOREM 3.2. *For every $h \in (0,1]$, for every $l_h > 0$, $\mathrm{c}_J > 0$, $r \geq 2$, $\epsilon > 0$, for $(\boldsymbol{\phi}_{n,h}, \mathbf{T}_{n,h}, \mathrm{q}_{1-\alpha,n})$ equal either to $(\boldsymbol{\phi}^{\mathrm{hom}}_{n,h}, \mathbf{T}^{\mathrm{hom}}_{n,h}, \mathrm{q}^{\mathrm{hom}}_{1-\alpha,n,h})$ or to $(\boldsymbol{\phi}^{\mathrm{iid}}_{n,h}, \mathbf{T}^{\mathrm{iid}}_{n,h}, \mathrm{q}^{\mathrm{iid}}_{1-\alpha,n,h})$:*

$$\forall \alpha, \beta > 0, \exists C > 0, \forall \mathrm{c} \geq C, \limsup_{n\to\infty} \alpha_n(\boldsymbol{\phi}_{n,h}) + \beta_n(\boldsymbol{\phi}_{n,h}, \mathrm{c}r_n) \leq \alpha + \beta, \tag{3.22}$$

*where $\alpha_n(\boldsymbol{\phi}_{n,h}) = \mathbb{P}_{\mu_0}(\mathbf{T}_{n,h} > \mathrm{q}_{1-\alpha,n})$ and $\beta_n(\boldsymbol{\phi}_{n,h}, \mathrm{c}r_n) = \sup_{\mathbb{\mu}_n\in\mathrm{H}_1(\mathrm{c}r_n)} \mathbb{P}_{\mathbb{\mu}_n}(\mathbf{T}_{n,h} \leq \mathrm{q}_{1-\alpha,n})$ is defined for several alternatives $\mathrm{H}_1(\mathrm{c}r_n)$:*

1. $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_1, h, \epsilon n^{-\frac{1}{2}}, \mathrm{c}n^{-\frac{1}{2}}\right)$, *for* $\boldsymbol{\phi}_{n,h} = \boldsymbol{\phi}^{\mathrm{hom}}_{n,h}$, *with* $r_n = n^{-\frac{1}{2}}$,
2. $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_2, h, \epsilon n^{-\frac{1}{r}}, \mathrm{c}n^{-\frac{1}{r}}\right)$, *for* $\boldsymbol{\phi}_{n,h} = \boldsymbol{\phi}^{\mathrm{iid}}_{n,h}$, *with* $r_n = n^{-\frac{1}{r}}$,
3. $\mathrm{H}_1\left(\mu_0, \mathcal{W}_1, l_h, \mathrm{c}n^{-\frac{1}{2}}\right)$, *for* $\boldsymbol{\phi}_{n,h} = \boldsymbol{\phi}^{\mathrm{hom}}_{n,h}$, *with* $r_n = n^{-\frac{1}{2}}$,
4. $\mathrm{H}_1\left(\mu_0, \mathcal{W}_2, l_h, \mathrm{c}n^{-\frac{1}{4}}\right)$, *for* $\boldsymbol{\phi}_{n,h} = \boldsymbol{\phi}^{\mathrm{iid}}_{n,h}$, *with* $r_n = n^{-\frac{1}{4}}$,
5. $\mathrm{H}_1\left(\mu_0, \mathcal{W}_2, l_h, \mathrm{c}_J, \mathrm{c}n^{-\frac{1}{2}}\right)$, *for* $\boldsymbol{\phi}_{n,h} = \boldsymbol{\phi}^{\mathrm{iid}}_{n,h}$, *with* $r_n = n^{-\frac{1}{2}}$,

*where all alternatives are defined in Section 3.1.*

The proof of Theorem 3.2 is available in Section S.5.3.2 in Brécheteau (2025).

3.2.3. *A lower bound on the separation rate for i.i.d. alternatives.* The following Theorem 3.3 states that no test for $\mathrm{H}_0$ against i.i.d. alternatives have a separation rate faster than $n^{-1}$.

THEOREM 3.3. *For every $h \in (0,1]$, for every sequence of $\{0,1\}$-valued measurable functions $(\phi_n)_{n\in\mathbb{N}}$ on $\mathscr{P}_n(\mathscr{X})$:*

$$\forall \gamma \in (0,1), \exists C > 0, \forall \mathrm{c} \leq C, \liminf_{n\to\infty} \alpha_n(\phi_n) + \beta_n(\phi_n, c) \geq 1-\gamma,$$

*where* $\alpha_n(\phi_n) = \mathbb{P}_{\hat{\mathbb{\mu}}_{0,n}}(\phi_n(\hat{\boldsymbol{\mu}}_{0,n}) = 0)$ *and where, for* $p \in \{1,2\}$,

$$\beta_n(\phi_n, c) = \sup_{\hat{\mathbb{\mu}}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\, \mathcal{W}_p(s_h(\mu_0), s_h(\mu)) \geq \frac{c}{n}} \mathbb{P}_{\hat{\boldsymbol{\mu}}_n \sim \hat{\mathbb{\mu}}_n}(\phi_n(\hat{\boldsymbol{\mu}}_n) = 1).$$

The proof of Theorem 3.3 is available in Section S.5.3.3 in Brécheteau (2025). It follows the proofs arguments for Theorem 1 in Lacour and Pham Ngoc (2014), based on Ingster (1993); Tsybakov (2009).

**4. Numerical illustrations.** In this part, we numerically investigate the performance of our two families of tests. The tests are implemented in the Julia package Brécheteau (2024). The inclusion of R and Python functions is done with the Julia packages `RCall` and `PyCall`. The numerical implementations specific to this paper are done in Jupyter notebooks, in the examples file of the Julia package Brécheteau (2024).

In Section 4.1, we start with a discussion about the selection of the regularity parameter $h$ for our tests $(\boldsymbol{\phi}^{\mathrm{iid}}_{n,h})_{0\leq h\leq 1}$. In Section 4.2, we compare our tests $(\boldsymbol{\phi}^{\mathrm{iid}}_{n,h})_{0\leq h\leq 1}$ and $(\boldsymbol{\phi}^{\mathrm{hom}}_{n,h})_{0\leq h\leq 1}$ to some tests available in the litterature for the circle $\mathbb{S}^1$, the sphere $\mathbb{S}^2$, the flat torus $\mathbb{T}^2$ and the Grassmannian $\mathbb{G}(2,4)$, for i.i.d. samples. In particular, the numerical illustrations enhance that our tests almost behave as the best tests for $h$ close to 1 for unimodal alternatives, whereas our tests significantly outperform existing tests of the litterature for $h$ close to 0 for balanced multimodal alternatives. In Section 4.3, we investigate the opposite behaviour of the two families of tests $(\boldsymbol{\phi}^{\mathrm{hom}}_{n,h})_{0\leq h\leq 1}$ and $(\boldsymbol{\phi}^{\mathrm{iid}}_{n,h})_{0\leq h\leq 1}$ on samples of dependent points, generated from a regular grid, with noise. In particular, for a small amount of noise, the tests $(\boldsymbol{\phi}^{\mathrm{hom}}_{n,h})_{0\leq h\leq 1}$ never reject $\mathrm{H}_0$, since the points are uniformly spread on the space, whereas the tests $(\boldsymbol{\phi}^{\mathrm{iid}}_{n,h})_{0\leq h\leq 1}$ always reject $\mathrm{H}_0$, since the data points are dependent. Finally, in Section 4.4, we propose an application of our statistical tests to shape analysis. Using the procedure of Buet, Leonardi and Masnou to approximate normal vectors or tangent spaces of surfaces and curves, we test uniformity of the distribution of the normal vectors and of the tangent spaces (in $\mathbb{S}^2$ and $\mathbb{G}(2,3)$) of the sphere and the Bunny of Stanford, two surfaces in $\mathbb{R}^3$. This method based on (barycenters of) DTM signatures of normal and tangent spaces distributions is a successful and promising attempt to compare shapes. In Section 4.4 of Brécheteau (2025), we also consider shapes in $\mathbb{R}^2$ and in $\mathbb{R}^4$.

4.1. *Selection of the regularity parameter: multiple testing procedures and aggregation of tests.* Each test defined in Section 3.1 test $\mathrm{H}_0$"$\mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}$" against the alternative $\mathrm{H}_1$"$\mathbb{\mu}_n \neq \hat{\mathbb{\mu}}_{0,n}$". More specifically, tests $\boldsymbol{\phi}_h$ are powerful against alternatives of type $\mathrm{H}_{1,h}$"$s_h(\mu) \neq s_h(\mu_0)$". Since $\mu_0$ is determined by the whole family of signatures $(s_h(\mu_0))_{h\in(0,1)}$ according to Proposition 2.3 and by continuity of $h \mapsto s_h(\mu)$ as a consequence of the continuity of $h \mapsto \mathrm{d}_{\mu,h}$ defined in (2.1), it would make sense to consider a grid of regularity parameters $(h_i)_{i\in I}$ and to reject $\mathrm{H}_0$ if one of the tests $(\boldsymbol{\phi}_{h_i})_{i\in I}$ rejects $\mathrm{H}_0$. Such a procedure would provide an adaptative test of uniformity that would adapt to any alternatives of $\mathrm{H}_0$. Intuitively, by choosing a small parameter $h_i$, we aim at detecting local variations of the density, whereas with large parameters $h_i$, we aim at detecting lack of global symmetry in the density. This kind of behaviour was noticed in Gretton et al. (2012) for kernel density-based statistical two-sample testing. The critical values should then be slightly modified to keep a test with the correct level, this is the principle of the Bonferroni and Benjamini-Hochberg multiple testing procedures. We implement both of these procedures, as well as the aggregation of tests procedure in Schrab et al. (2023); Fromont and Laurent (2006) that selects a single parameter $h_i$ in $(h_i)_{i\in I}$ based on all test statistics. All these methods are recalled in Brécheteau (2025), in Section S.3.1.1. We also implement a new aggregation method that we detail below.

4.1.1. *Presentation of the new alternative aggregation of tests procedure.* Let $\alpha \in (0,1)$ be the nominal level. We first select the parameter $h_{i_0}$ that is most likely to provide the higgest power, that is, the parameter $h$ for which the p-value of the test is the smallest. Then, for the test to have the proper nominal level $\alpha$, we compare the statistic $\mathbf{T}_{n,h_{i_0}}$ to the quantile of $\mathbf{T}_{n,h_{i_0}}$ conditionally to the selection of $i_0$, under the null hypothesis, as follows. Let $\left(\mathbf{p}^{obs}_{n,h_i} = \mathbb{P}_{\hat{\mu}_{0,n}}\left(\mathbf{T}_{n,h_i} > \mathbf{T}^{obs}_{n,h_i}\right)\right)_{i\in I}$ be the p-values of a family of tests $(\phi_{n,h_i})_{i\in I}$ with observed statistics $\left(\mathbf{T}^{obs}_{n,h_i}\right)_{i\in I}$ based on data. Let $i_0 = \arg\min_{i\in I} \mathbf{p}^{obs}_{n,h_i}$. We define the adaptive test $\phi_{n,\alpha} = \mathbb{1}_{\mathbf{T}_{n,h_{i_0}} > \mathrm{q}_{1-\alpha,n,h_{i_0}|i_0}}$, where $\mathrm{q}_{1-\alpha,n,h_{i_0}|i_0}$ is the $1-\alpha$ quantile of the distribution of $\mathbf{T}_{n,h_{i_0}}$ conditionally to $\arg\min_{i\in I} \mathbf{p}_{n,h_i} = i_0$.

4.1.2. *Comparison of the methods to select the best parameters.* We apply the classical method of aggregation, the alternative method of aggregation defined above, the Benjamini-Hochberg procedure and the Bonferroni procedure to select parameters $h_i$ in $(h_i)_{i\in I} = \left(\frac{i}{10}\right)_{1\le i\le 10}$. For several sampling models on $\mathbb{S}^1$, we generated 10000 samples of size $n = 100$. For the two first methods, we count the proportion of selection of each parameter $(h_i)_{i\in I}$. For the last two procedures, since several parameters can be selected at the same time (the $h_i$ such that the $i$-th test rejects $\mathrm{H}_0$), for each parameter $h_i$, we count the number of rejections, and we divide these numbers by the total number of rejections, c.f. Figure 2.

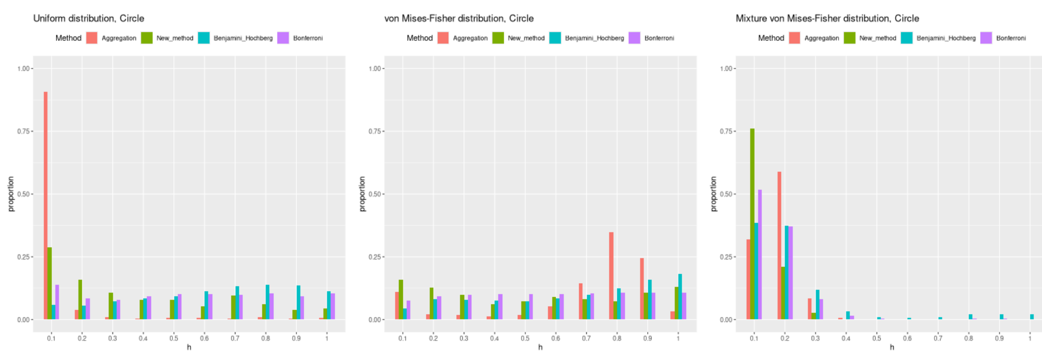


FIG 2. *Selection of the parameters h.*

We consider the following sampling methods on $\mathbb{S}^1$: the von Mises-Fisher distribution with parameter $\kappa = 0.5$ and a mixture of 4 von Mises-Fisher distributions with centers on a regular polytope, with parameter $\kappa = 10$. The quantiles of the tests are computed by a Monte-Carlo procedure with 10000 samples of size $n = 100$. The powers estimations for the four procedures are available in Table 1.

| Method | Class. Aggregation | Alt. Aggregation | Benjamini-Hochberg | Bonferroni |
|---|---|---|---|---|
| Uniform | 0.0528 | 0.0533 | 0.0361 | 0.0111 |
| von Mises-Fisher | 0.8531 | 0.8337 | 0.8447 | 0.6649 |
| Mixture von Mises-Fisher | 0.9385 | 0.9331 | 0.7771 | 0.7512 |

TABLE 1
*Power comparison for tests of nominal level 0.05, on the circle* $\mathbb{S}^1$.

We observe that the power of the classical aggregation method and the alternative aggregation method are similar. The classical aggregation method is yet faster to calibrate.

Notice that if the Benjamini-Hochberg is not proved correct in this context since p-values are not independent, its type I error is lower than $5\%$, therefore it is valid on our examples. The Benjamini-Hochberg procedure and the Bonferroni procedure lead to a type I error far smaller than $5\%$. Therefore, such procedures have a lower power. Such methods will not be used for testing uniformity, but only for parameters selection. Moreover, according to Figure 2 these methods confirm that on $\mathbb{S}^1$ small parameters $h_i$ lead to more powerful tests to detect mixtures of von Mises-Fisher distributions, whereas large parameters $h_i$ lead to more performant tests to detect single von Mises-Fisher alternatives. In Brécheteau (2025), in Section S.3.1.2, we consider the same numerical procedures on $\mathbb{S}^2$, $\mathbb{T}^2$ and $\mathbb{B}$, and observe the same behaviours.

4.2. *Comparison to classical tests of uniformity.* In the litterature, numerous tests of uniformity have been designed, for specific homogeneous spaces, the circle, the sphere, the flat torus and the Grassmannian. In this section, we compare the performance of our tests to some of these classical tests.

4.2.1. *On the circle $\mathbb{S}^1$ and on the sphere $\mathbb{S}^2$.* We compare our tests $(\phi_{n,h}^{\mathrm{hom}})_{0\leq h\leq 1}$ (`homogen`) and $(\phi_{n,h}^{\mathrm{iid}})_{0\leq h\leq 1}$(`iidness`), for parameters $h \in [0.1, 0.2, 0.5, 1]$, as well as the alternative aggregation methods (`hom_select` and `iid_select`), to the tests of uniformity available in the R package `sphunif` in García-Portugués and Verdebout (2024) and described in the overview García-Portugués and Verdebout (2018).

For the unit circle $\mathbb{S}^1$, we consider samples of size $n = 100$ from the von Mises-Fisher distribution with parameter $\kappa = 0.5$, from a mixture of von Mises-Fisher distributions with parameter $\kappa = 10$, with 4 centers on a regular polytope and from the uniform distribution. For the sphere $\mathbb{S}^2$, we consider samples of size $n = 100$ from the von Mises-Fisher distribution with parameter $\kappa = 0.5$, from a mixture of von Mises-Fisher distributions with parameter $\kappa = 10$, with 6 centers on a regular polytope and from the uniform distribution. In Figure 3, we compute the percentage of rejection of $\mathrm{H}_0$ at the $5\%$ nominal level in 1000 tests replications for samples of size $n = 100$. As expected, for uniformly distributed samples, the power is approximatively equal to $5\%$, meaning that all of the tests have the correct nominal level.

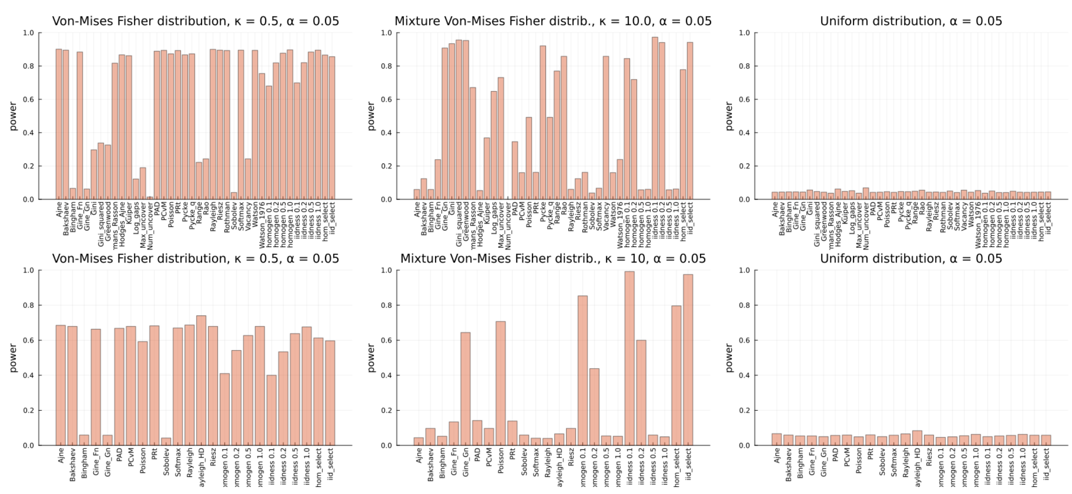


FIG 3. *Powers on the circle (top) and on the sphere (bottom)*

Our methods for large parameter $h$ have the same performances as the best tests for unimodal alternatives. Whereas our methods for small parameter $h$ strongly outperfrom existing methods for multimodal alternatives. This is particularly the case for instance for the mixture of 6 von Mises-Fisher distributions alternative. Some of our tests have a power equal to 1, whereas other tests have a power inferior to 0.7). Our alternative aggregation procedure (2 right-columns) have a power almost as good as the tests with the best parameters in both contexts, and still have the correct level.

In Brécheteau (2025), in Figure S.7, we also compute the p-values of the tests on regular samples on $\mathbb{S}^1$ and $\mathbb{S}^2$. Roughly, all tests, including our tests $\phi_{n,h}^{\mathrm{hom}}$, have a p-value equal to 1, meaning that the hypothesis of uniformity is strongly accepted, except for our tests $\phi_{n,h}^{\mathrm{iid}}$ (and for $\mathbb{S}^1$, `Log_gasps` and `Num_uncover`) for which the p-value is 0, or almost 0, meaning that the assumption of uniformity is strongly rejected. Contrary to most of the other tests, our tests $\phi_{n,h}^{\mathrm{iid}}$ detect non iidness. In Section 4.3, we investigate more deeply the difference of behaviour of our two families of tests.

4.2.2. *On the flat torus* $\mathbb{T}^2$. We compare our tests $(\phi_{n,h}^{\mathrm{iid}})_{0\leq h\leq 1}$ with parameters $h \in [0.1, 0.3, 0.5, 0.7, 0.9]$, to three existing tests of uniformity. First, we compare to the two-dimensional extension of the Kolmogorov-Smirnov two-sample test in Fasano and Franceschini (1987), implemented in the R package `fasano.franceschini.test`, Puritz (2023). We consider the two-sample test and compare to a uniformly distributed sample of size 1000. Then, we compare to the transport-based test of uniformity, based on the $L_2$-Wasserstein disance, of Hallin, Mordant and Segers (2021), using the R package `transport`. Finally, we compare to the Sobolev test of type Rayleigh, as described in Jupp (2009), for $k = 1$.

We consider mixtures of normal distributions, with centers in a regular grid of $s \times s$ points with $s \in \{1, 2, 3\}$, with standard deviations in $[0.02, 0.05, 0.1, 0.15, 0.2, 0.25]$ (in $[0.15, 0.3, 0.5, 0.7, 1]$ for the unimodal distribution). We compute the percentage of rejection of $\mathrm{H}_0$ at the $5\%$ nominal level in 100 tests replications for samples of size $n = 10$. The powers are displayed in Figure 4. Our tests perform well for unimodal alternatives, and outperform existing procedures for multimodal alternatives.

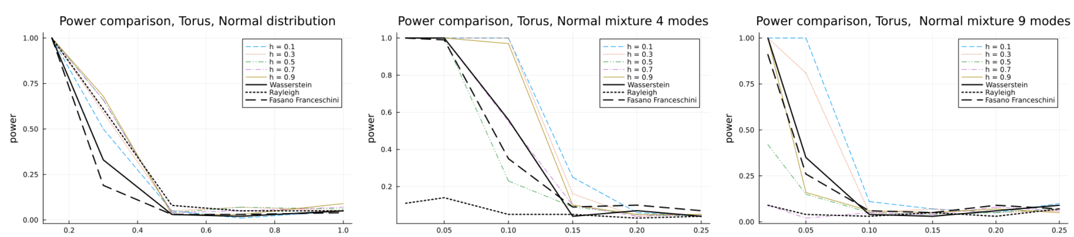


FIG 4. *Power comparison, flat torus* $\mathbb{T}^2$

4.2.3. *On the Grassmannian* $\mathbb{G}(2,4)$. We compare our tests $(\phi_{n,h}^{\mathrm{iid}})_{0\leq h\leq 1}$ with parameters $h \in [0.1, 0.3, 0.5, 0.7, 0.9]$ to the function `grassmann.utest` of the R package `Riemann`, that test uniformity of samples in the Grassmannian $\mathbb{G}(2,4)$. This method is based on the Bingham's test, Chikuse (2003); Mardia and Jupp (2000).

*4.2.3.1. Method 1:.* For each standard deviation $\sigma$ in $[0, 0.5, \ldots, 4.5, 5]$, we generate 1000 samples of size $n = 100$, where each sample point is given by the two first eigenvectors of the covariance matrix of 50-samples $(s_i + X_i)_{1\leq i\leq 50}$, where $(s_i)_{1\leq i\leq 50} \in (\mathbb{R}^4)^{50}$ are the 4 first coordinates of the 50 first points of the `iris` R dataset, and $(X_i)_{1\leq i\leq 50}$ are i.i.d. 50-samples with normal distribution $\mathscr{N}(0, \sigma^2)$. Notice that for small values of $\sigma$, the distribution of the sample points will not be uniform on $\mathbb{G}(2,4)$ since they will all be close to the two first eigenvectors of the `iris` R dataset, whereas the signal of the data set gets erased by the noise when $\sigma$ gets large. Therefore, we expect a power close to 1 for small values of $\sigma$, and a power close to $0.05$ (the level of the test) for larger values of $\sigma$.

*4.2.3.2. Method 2:.* For each parameter $\sigma$ in $[0.01, 0.03, 0.05, 0.07, 0.1, 0.3, 0.5, 1]$, we consider a mixture of 6 normal distributions in $\mathbb{G}(2,4)$ with centers given by $(e_1, e_2)$, $(e_1, e_3)$, $(e_1, e_4)$, $(e_2, e_3)$, $(e_2, e_4)$, $(e_3, e_4)$ (for $e_1, e_2, e_3, e_4$, the vectors of the canonical basis), and with standard deviation $\sigma$. The probability of each component is given by $\frac{1}{6}$. Samples are generated with the package `Distribution.jl`, in particular, with the functions `Grassmann`, `rand` to generate random normal vectors on tangent spaces, and with the exponential function `exp`. Notice that the largest the standard deviation is, the closest the distribution is to the uniform distribution.

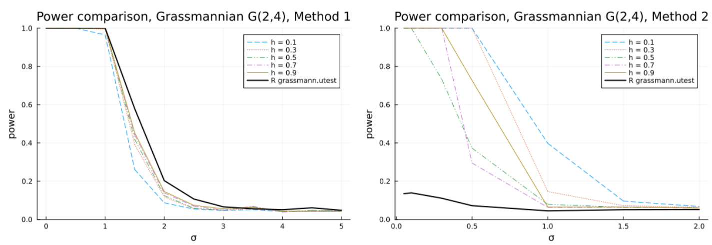


FIG 5. *Power comparison, Grassmannian $\mathbb{G}(2,4)$ manifold*

According to Figure 5, our tests of iidness are almost as performant as the function `grassmann.utest` of the R package `Riemann`, for the unimodal alternative, but completely outperform the function `grassmann.utest` for the multimodal alternative. As expected, for the first case, the best parameters $h$ are the largest, whereas for the second case, the best parameters $h$ are the smallest, since they catch local variations.

4.3. *Investigating the power of tests for non i.i.d. samples.* Tests of homogeneity $(\phi^{\mathrm{hom}}_{n,h})_h$ have a null power under alternatives $\mathbb{\mu}_n$ supported on the subset of $\mathscr{P}_n(\mathscr{X})$, which signatures are as close as possible to $s_h(\mu_0)$:

PROPOSITION 4.1. *Let $h \in (\frac{1}{n}, 1]$. Assume that $\mathscr{X}$ is not finite. Let $\mathscr{P}^{\mathrm{opt}}_{n,h}(\mathscr{X})$ be the set of measures supported on $n$ points, which signature is the closest to the signature of the uniform measure $\mu_0$:*

$$\mathscr{P}^{\mathrm{opt}}_{n,h}(\mathscr{X}) = \arg \min_{\{\mu_n \in \mathscr{P}_n(\mathscr{X})\}} \mathcal{W}_1\left(s_h(\mu_n), s_h(\mu_0)\right).$$

*The set $\mathscr{P}^{\mathrm{opt}}_{n,h}(\mathscr{X})$ is not empty. Moreover, for any measure $\mathbb{\mu}_n$ supported on $\mathscr{P}^{\mathrm{opt}}_{n,h}(\mathscr{X})$, the power of the test $\phi^{\mathrm{hom}}_{n,h}$ is equal to 0:*

$$\mathbb{P}_{\mathbb{\mu}_n}(\mathbf{T}^{\mathrm{hom}}_{n,h} > \mathrm{q}^{\mathrm{hom}}_{1-\alpha,n,h}) = 0.$$

The proof of Proposition 4.1 is available in Section S.5.3.4. A direct consequence of Proposition 4.1 and Proposition 2.1 is that any measure $\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$, close enough to $\mathscr{P}^{\mathrm{opt}}_{n,h}(\mathscr{X})$, in terms of $\mathcal{W}_{1,\mathcal{W}_2}$, has a power lower than $\alpha$. Surprisingly, supports of measures in $\mathscr{P}^{\mathrm{opt}}_{n,h}(\mathscr{X})$ are not necessarily regular grids. For instance, for the circle $\mathbb{S}^1$, $n = 4$ and $h =$

$\frac{1}{2}$, we get that $\mathscr{P}^{\mathrm{opt}}_{4,\frac{1}{2}}(\mathbb{S}^1) = \left\{\frac{1}{4}\sum_{i=1}^{4}\delta_{x_i},\ \forall i \in \{1,\dots,4\},\ \min_{j\in\{1,\dots,4\}\setminus\{i\}} \mathrm{d}(x_i,x_j) = \frac{\pi}{\sqrt{6}}\right\}$, since for such measures $\mu_4 \in \mathscr{P}^{\mathrm{opt}}_{4,\frac{1}{2}}(\mathbb{S}^1)$, $s_{\frac{1}{2}}(\mu_0) = s_{\frac{1}{2}}(\mu_4) = \delta_{\frac{\pi}{\sqrt{12}}}$. Although the signature of the grid $s_h(\mu_n)$ with $\mu_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{\left(\cos\left(2\pi\frac{k}{n}\right),\sin\left(2\pi\frac{k}{n}\right)\right)}$, is at a distance to the signature of $\mu_0$, of order $\frac{1}{n}$: $\mathcal{W}_1(s_h(\mu_n), s_h(\mu_0)) \leq \mathcal{W}_2(s_h(\mu_n), s_h(\mu_0)) \leq \frac{1}{\sqrt{h}}\mathcal{W}_2(\mu_n,\mu_0) \leq \frac{1}{\sqrt{h}}\frac{\pi}{n}$, using Proposition 2.1 and considering the transport plan that sends each Voronoi cells to the center, in the decomposition of $\mathbb{S}^1$ into Voronoi cells with centers in $\mathrm{Supp}(\mu_n)$. For $\mathbb{S}^1$ at least, signatures on uniform grids seem to approach quite fast the true signature $s_h(\mu_0)$, in comparison to empirical signatures that we expect to approximate the true signature at a parametric rate. Therefore, we expect the signature of the grid to be closer to the signature of the uniform distribution than to the barycenter signature, compared to an empirical signature. Therefore, we expect that tests $\phi^{\mathrm{iid}}_{n,h}$ reject the null hypothesis $\mathrm{H}_0$ under such an alternative. This intuition is confirmed below, numerically, for $\mathbb{S}^2$.

We consider a grid on the sphere $(s_1,\dots,s_{98})$, of size 98, based on Buet, Leonardi and Masnou implementation. For each parameter $\kappa$ in $[0, 0.5, 1, 3, 5, 10, 20, 50, 100, 200, 500]$, we generate 10000 samples of size 98, based on the following procedure : $(X_{1,\kappa},\dots,X_{98,\kappa})$ are independent random variables, with, for $1 \leq j \leq 98$, $X_{j,\kappa}$ generated according to the von Mises-Fisher distribution on the sphere, with parameter $\kappa$ and center $s_j$. For $\kappa = 0$, $(X_{1,\kappa},\dots,X_{98,\kappa})$ is a sample from the uniform distribution on the sphere, while for $\kappa = +\infty$, $(X_{1,\kappa},\dots,X_{98,\kappa}) = (s_1,\dots,s_{98})$ coincides with the grid.

We compare the power of the two families of tests in Figure 6. The tests have a level $\alpha = 0.05$. Their quantiles are estimated by a Monte-Carlo procedure, after 10000 replications. The two families of tests start with a power of 0.05, which corresponds to the level of the tests. The tests of iidness have a power that converges to 1, while the tests of homogeneity have a power that converges to 0, when $\kappa$ goes to $+\infty$. Indeed, samples of points will be closer and closer to a regular grid, therefore we loose independence of sample points (power converges to 1 for iidness tests), but we improve the homogeneity of the sample, since grids are more uniformly spread on the sphere than an i.i.d. sample from the uniform distribution (power converges to 0 for homogeneity tests).

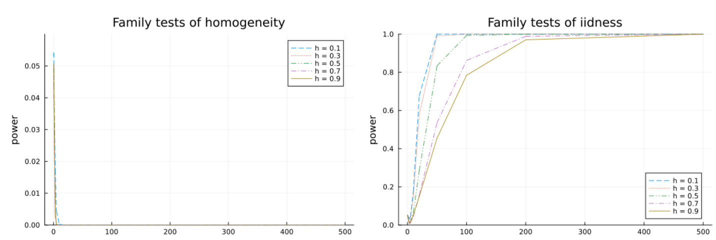


FIG 6. *Power comparison of families of tests*

4.4. *Application to shape analysis.* In this section, we consider sets of points on the sphere (34686 points) from the python notebook Buet, Leonardi and Masnou, and the Stanford bunny (34835 points), from the Stanford University Computer Graphics Laboratory, http://graphics.stanford.edu/data/3Dscanrep/. To all points, we associate an estimator of the normal vector (and its opposite) in $\mathbb{S}^2$ and of the tangent space in $\mathbb{G}(2,3)$, using Buet and Rumpf (2022); Buet, Leonardi and Masnou (2022) and its Python implementation, in Buet, Leonardi and Masnou. The samples and distributions of vectors are displayed in Figure 7.

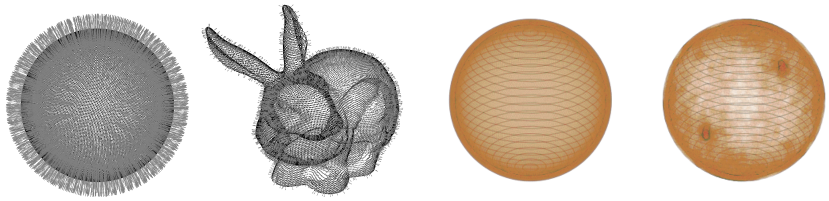

FIG 7. *Point clouds (left) and normal vectors (right)*

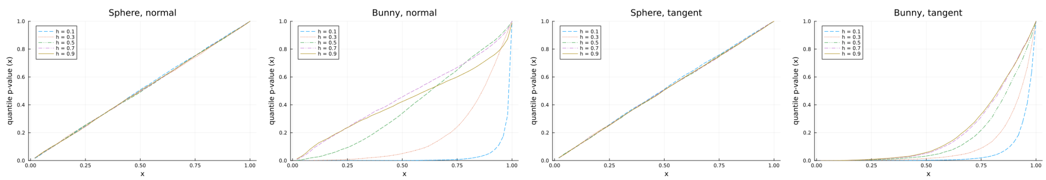


FIG 8. *Distribution of p-values for the iidness tests, normal directions (left), tangent spaces (right)*

The density of the normal vectors and of the tangent spaces is uniform for the sphere but not for the bunny. This is confirmed by Figure 8 where we plot 10000 sorted p-values of iidness tests based on 100-samples, with parameter $h$ varying in $[0.1, 0.3, 0.5, 0.7, 0.9]$. The p-values are aligned with the line $x \mapsto x$ on $[0, 1]$ for the sphere, that illustrates the uniformity of the measure, unlike for the bunny. Small values of $h$ provide more powerful tests. These illustrations enhance that the DTM-signatures or barycenter signatures for normal vector or tangent spaces provide new shape descriptors, and may be used for shape comparison, clustering, or even for testing equality of shapes, as alternatives to Mémoli (2011); Osada et al. (2002).

**5. Conclusions and Perspectives.** In this paper, we have defined two families of statistical tests to test that a sample of $n$ points is uniform on some homogeneous space $\mathscr{X}$. We provided theoretical results for the consistency of the tests, that come with separation rates. We illustrated the performance of these tests on simulated samples on the circle, the sphere, the flat torus, the Grassmannian but also on the Bolza surface, in Brécheteau (2025). We used classical aggregation of tests procedures and developped a new one, to take advantage of the best regularity parameter $h$, depending on the alternative, making the tests adaptative, despite a slight loss of power. For large parameters, the tests compare to the best tests against unimodal alternatives, whereas for small parameters, the tests outperform existing tests, for multimodal alternatives. We also investigated the difference of behaviour of the two tests under non i.i.d. alternatives.

In a future work, we will use the tests $(\phi^{\mathrm{iid}}_{n,h})_{h\in(0,1)}$ to validate new sampling procedures based on dynamical systems theory to generate samples that behave as i.i.d. uniform samples. Besides, the test statistics $(\mathbf{T}^{\mathrm{hom}}_{n,h})_{h\in(0,1)}$ can be used to detect samples that are more homogeneous than i.i.d. uniform samples, if rejecting when the statistic is small enough. Both tests could also be used for goodness-of-fit testing, after transporting the sample through the optimal transport plan to the uniform distribution. Finally, in Section 4.4, we enhanced the possibility for the statistical tests of uniformity, together with the procedure to estimate normal vectors and tangent spaces of Buet and Leonardi (2016), to be used to reject uniformity of normal vectors or tangent spaces. DTM signatures based on normal vectors or tangent spaces distributions are actually new shape descriptors that may be used for shape comparison, clustering or testing.

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