SUPPLEMENT TO "TWO DISTANCE-BASED FAMILIES OF STATISTICAL TESTS OF UNIFORMITY FOR PROBABILITY MEASURES ON HOMOGENEOUS COMPACT POLISH SPACES"

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In this supplement of Brécheteau (2025), we come back to the resuts of Christensen (1970) relative to the unicity of the uniform measure and to the determination of measures by its values on balls. Then we provide additional discriminative results for the DTM-signatures, compute the signature of the uniform distribution for the torus, the sphere and the Bolza surface and define and discuss properties of median signatures. We provide additional numerical illustrations as well as technical details for simulations, including details on the Monte Carlo estimators of Wasserstein signatures, details on the generation of samples from the uniform distribution on hyperbolic spaces and from (mixtures of) normal distributions on general Riemanian manifolds. We finally give the proofs of the results of the main paper Brécheteau (2025) and of this supplement.

S.1. On the unicity of the uniform measure and on the determination of measures by their values on balls. The question to know whether or not measures are determined by their values on balls is non trivial Christensen (1980). Measures are determined by their values on balls in Banach spaces Preiss and Tišer (1991) after Hoffmann-Jø rgensen (1975); Dinger (1986), and more specifically in Euclidean spaces Zelený (2000), and for more general spaces Buet and Leonardi (2016), but not on any compact metric space Davies (1971); Keleti and Preiss (2000). Counter-example in Davies (1971) uses the link with the problem of finding for every $\epsilon, \delta > 0$, disjointed closed balls $S_1, \ldots, S_N$ with radius not exceeding δ and such that $\mu(\mathscr{X} \setminus \bigcup_{j=1}^N S_j) < \epsilon$.

The unicity of uniform and h_0-uniform Borel probability measures, when it exists. This is given by Theorem S.1.1.

THEOREM S.1.1 (Christensen (1970)). *For any compact metric space $(\mathscr{X}, \mathrm{d})$, an h_0-uniform measure μ_0, if it exists, is unique. Indeed, for every $\mu, \nu \in \mathscr{P}(\mathscr{X})$, if μ and ν coincide on balls, i.e. satisfy (1.1) or (1.2) from Brécheteau (2025) for some $h_0 > l_0$, then $\mu = \nu$.*

Moreover, assuming the existence of an h_0-uniform measure μ_0, if $\nu \in \mathscr{P}(\mathscr{X})$ is such that for some sequence $(r_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_+^*$ converging to 0,

$$\forall x \in \mathscr{X}, \forall n \in \mathbb{N}, \nu(\mathrm{B}(x, r_n)) = \mu_0(\mathrm{B}(x, r_n)), \tag{S.1.1}$$

then, $\nu = \mu_0$.

The proof of Theorem S.1.1 is available in Section S.6.1.1. This result is a consequence of the more general result of Christensen (1970), that states that for metric spaces for which there exists a quasi-uniform probability measure, measures are determined by their values on balls with radius smaller than a fixed radius. We write the proof when $(\mathscr{X}, \mathrm{d})$ is compact, but the result of Christensen (1970) is more general. This is Theorem S.1.2.

DEFINITION S.1.1. *A quasi-uniform Borel probability measure μ_0 on a compact metric space $(\mathscr{X}, \mathrm{d})$ is a Borel probability measure so that:*

$$\lim_{\epsilon \to 0} \sup_{x,y \in \mathscr{X}} \left| \frac{\mu_0\left(\mathrm{B}(x,\epsilon)\right)}{\mu_0\left(\mathrm{B}(y,\epsilon)\right)} - 1 \right| = 0. \tag{S.1.2}$$

THEOREM S.1.2 (Christensen (1970)). *If $(\mathscr{X}, \mathrm{d})$ is a compact Polish space for which there exists a quasi-uniform Borel probability measure μ_0, then, for every Borel probability measures μ and ν, for every $\epsilon_0 > 0$,*

$$\mu(\mathrm{B}(x,\epsilon)) = \nu(\mathrm{B}(x,\epsilon)), \forall x \in \mathscr{X}, \forall 0 < \epsilon < \epsilon_0 \Rightarrow \mu = \nu. \tag{S.1.3}$$

The proof of Theorem S.1.2 is inspired from the proof of Theorem S.1.1. It is available in Section S.6.1.2. Consequently, on such metric spaces, measures with the same DTM functions for every parameter $h \in [0, h_0]$ for some $h_0 > 0$, coincide.

S.2. Signatures - Supplement.

S.2.1. *Additional discriminative properties of DTM-signatures.* We consider the set of values $h \in [0,1]$ for which the DTM-signature with parameter h characterises the uniform distribution,

$$\mathscr{H}(\mathscr{X}) := \{h \in [0,1], \forall \mu \in \mathscr{P}(\mathscr{X}), s_h(\mu) = s_h(\mu_0) \Rightarrow \mu = \mu_0\}. \tag{S.2.1}$$

In general, for $\mathscr{M} \subset \mathscr{P}(\mathscr{X})$, we define

$$\mathscr{H}(\mathscr{M}) := \{h \in [0,1], \forall \mu \in \mathscr{M}, s_h(\mu) = s_h(\mu_0) \Rightarrow \mu = \mu_0\}, \tag{S.2.2}$$

so that $\mathscr{H}(\mathscr{X}) = \mathscr{H}(\mathscr{P}(\mathscr{X}))$.

Tests for uniformity based on a DTM-signature with parameter $h \in \mathscr{H}(\mathscr{X})$ or $\mathscr{H}(\mathscr{M})$ are powerful, c.f. Section 3.1.1 in Brécheteau (2025). The set $\mathscr{H}(\mathscr{M})$ is not empty when $\mathscr{M}$ is a set of measures with support different to $\mathscr{X}$, as noticed in Proposition S.2.1 and Proposition S.2.2. This is also the case for discrete homogeneous compact sets, as noticed in Proposition S.2.3. However, in this case, the set $\mathscr{H}(\mathscr{X})$ does not coincide with $[0,1]$, as noticed in Proposition S.2.4.

It means that for measures with support different to $\mathscr{X}$, all DTM-signatures are discriminative, provided that the parameter h is small enough. For discrete spaces, this is also the case, for parameters h not too small.

Notice that some of these examples are inspired from Brécheteau (2019) that provide examples for which signatures are discriminative between two different metric measure spaces.

S.2.1.1. *On continuous spaces.* The DTM signature for small parameters h discriminate measures with support different to $\mathscr{X}$, from the uniform measure μ_0:

PROPOSITION S.2.1. *Let $\epsilon > 0$. For every $\mu \in \mathscr{P}(\mathscr{X})$, if $\mathrm{d_H}(\mathrm{Supp}(\mu), \mathscr{X}) \geq \epsilon$, then:*

$$\forall h \leq h(\epsilon),\ s_h(\mu) \neq s_h(\mu_0), \tag{S.2.3}$$

where $\mathrm{d_H}$ is the Hausdorff distance, so that

$$\mathrm{d_H}(A, \mathscr{X}) = \sup_{x \in \mathscr{X}} \inf_{y \in A} \mathrm{d}(x,y) = \sup_{x \in \mathscr{X}} \mathrm{d}(x, A) \tag{S.2.4}$$

and $h(\epsilon) = \mu_0(\mathrm{B}(x,\epsilon))$.

In particular, $\mathscr{H}(\{\mu \in \mathscr{P}(\mathscr{X}), \mathrm{d_H}(\mathrm{Supp}(\mu), \mathscr{X}) \geq \epsilon)\}) \supset [0, h(\epsilon)]$.

The proof of Proposition S.2.1 is available in Section S.6.2.1.

More precisely, for a measure with constant density with respect to μ_0 on its support, we get the following lower bound for the distance between its DTM-signature and the DTM-signature of μ_0.

PROPOSITION S.2.2. *Let $h\in(0,1)$ and $l\in(0,1)$. Let $\mu_l\in\mathscr{P}(\mathscr{X})$ be a Borel probability measure supported on $\mathrm{Supp}(\mu_l)\subsetneq\mathscr{X}$, with density $\frac{1}{\mu_0(\mathrm{Supp}(\mu_l))}1_{\mathrm{Supp}(\mu_l)}=\frac{1}{1-l}1_{\mathrm{Supp}(\mu_l)}$ with respect to μ_0. Let $A_{l,h}:=\{x\in\mathscr{X}\mid \mathrm{B}(x,r_{h(1-l)})\subset\mathrm{Supp}(\mu_l)\}$, with $r_{h(1-l)}=\delta_{\mu_0,h(1-l)}(x)$ for all $x\in\mathscr{X}$, the radius of a ball with μ_0-mass $h(1-l)$, as defined in (2.2). Then, for d_h and $\mathrm{d}_{h(1-l)}$ the constant values of the DTM to the uniform measure μ_0 with respective parameters h and $h(1-l)$, as defined in (2.5), we have that*

$$\mathcal{W}_1(s_h(\mu_l),s_h(\mu_0))\geq\frac{1}{1-l}\mu_0(A_{l,h})|\mathrm{d}_h-\mathrm{d}_{h(1-l)}|\sim_{l\to0}\mu_0(A_{l,h})C_hl,\tag{S.2.5}$$

where C_h is a non negative constant that depends on h only, equal to 0 if and only if $\mathscr{X}$ is discrete with $h<\frac{1}{|\mathscr{X}|}$, and where we recall that $\mu_0(A_{l,h})\in[0,1]$.

The proof of Proposition S.2.2 is available in Section S.6.2.2.

S.2.1.2. *On discrete spaces.* In the following, we denote by $0=d_1\leq d_2\leq\ldots\leq d_N$ the sorted distances of a point $x\in\mathscr{X}$ to other points in $\mathscr{X}$. We recall that this sequence does not depend on $x\in\mathscr{X}$, as noticed in Section 2.1.1 in Brécheteau (2025). Indeed, since μ_0 is uniform on $\mathscr{X}$:

$$\forall x\in\mathscr{X},\ \mu_0(\mathrm{B}_{x,r_1})=\mu_0(\{x\})=\frac{1}{|\mathscr{X}|}\tag{S.2.6}$$

for $r_1=\min\{\mathrm{d}(x,y),\,x,y\in\mathscr{X},x\neq y\}$. For every $r>r_1$, $\mu_0(\mathrm{B}_{x,r})\geq\frac{2}{|\mathscr{X}|}$, so that $\mathrm{d}(x,x_1(x))=r_1$ for every $x\in\mathscr{X}$. We conclude by induction. In particular, d_2 is the minimal distance between two distinct elements of $\mathscr{X}$ and $d_N=\mathscr{D}(\mathscr{X})$ is the diameter of $\mathscr{X}$.

PROPOSITION S.2.3. *If $\mathscr{X}$ is a discrete space with cardinality N, then $\mathscr{H}(\mathscr{X})$ is of non empty interior. In particular $\left[\frac{1}{N},\frac{k+1}{N}\right]\subset\mathscr{H}(\mathscr{X})$, where $k=|\{j\in\{2\ldots N\},\,d_j=d_2\}|$ is the number of points in $\mathscr{X}\backslash\{x\}$ at minimal distance to a point $x\in\mathscr{X}$.*

Moreover, for every $h\in\left[\frac{1}{N},\frac{k+1}{N}\right]$,

$$\begin{aligned}\mathcal{W}_1\left(s_h(\mu),s_h(\mu_0)\right)&\geq\frac{d_2}{N}\sum_{x,\,\mu(\{x\})<\frac{1}{N}}\left(\sqrt{1-\frac{\mu(\{x\})}{h}}-\sqrt{1-\frac{1}{Nh}}\right)\\&\geq\frac{d_2}{2Nh}\mathrm{d}_{\mathrm{TV}}(\mu,\mu_0).\end{aligned}$$

where $\mathrm{d}_{\mathrm{TV}}(\mu,\mu_0)=\frac{1}{2}\sum_{x\in\mathscr{X}}|\mu(\{x\})-\mu_0(\{x\})|$ denotes the Total Variation distance between μ and μ_0.

It implies that the L_1-Wasserstein distance to the signature of μ_0 is lower bounded by Wasserstein distances to μ_0:

$$\mathcal{W}_1\left(s_h(\mu),s_h(\mu_0)\right)\geq\frac{d_2}{2Nhd_N^2}\mathcal{W}_2^2(\mu,\mu_0)\tag{S.2.7}$$

and

$$\mathcal{W}_1\left(s_h(\mu),s_h(\mu_0)\right)\geq\frac{d_2}{2Nhd_N}\mathcal{W}_1(\mu,\mu_0).\tag{S.2.8}$$

The proof of Proposition S.2.3 is available in Section S.6.2.3.

Note that for discrete spaces, Proposition 2.3 in Brécheteau (2025) is a consequence of Proposition S.2.3.

DTM-signatures do not characterise uniform distributions for every values of $h \in [0,1]$.

PROPOSITION S.2.4. *If $\mathscr{X}$ is a discrete space with cardinality N, then $\mathscr{H}(\mathscr{X})^c$ is of non empty interior. In particular $\left[0, \frac{1}{N}\right) \subset \mathscr{H}(\mathscr{X})^c$.*

The proof of Proposition S.2.4 is available in Section S.6.2.4.

We conclude with additional examples on discrete spaces. For a discrete metric space with cardinality $N = 3$, symmetry properties imply that the distance to the second and third nearest neighbour in $\mathscr{X}$ are equal: $d_2 = d_3$. Therefore, according to Proposition S.2.3 and Proposition S.2.4, $\mathscr{H} = [\frac{1}{3}, 1]$. However, when $N \geq 4$, $\mathscr{H} \neq [\frac{1}{N}, 1]$ in general. Indeed, consider the metric space $\mathscr{X}$ of cardinality $N = 4$, so that $d_2 = d_3 = 1$ and $d_4 = \sqrt{2}$. Consider the measure μ that puts masses $\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}$ to elements of $\mathscr{X}$ so that the second and third nearest neighbours of each point have the same mass. Then, μ satisfies $s_1(\mu) = s_1(\mu_0) = \delta_1$, since $\frac{1}{6} + \frac{1}{6} + \sqrt{2}^2 \times \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \sqrt{2}^2 \times \frac{1}{6} = \frac{1}{4} + \frac{1}{4} + \sqrt{2}^2 \times \frac{1}{4} = 1 = 1^2$.

S.2.1.3. *Example of two different measures with the same barycenter signature.* In this section, we give the example of two different measures based on geodesics that do have the same non-trivial barycenter signature.

EXAMPLE S.2.1. *Let $\delta_{\mu_{x,n}}$ and $\alpha\delta_{\mu_{y,n}} + (1-\alpha)\delta_{\mu_{z,n}}$ be two measures in $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$, for some $\alpha \in (0,1)$; where $\mu_{x,n} = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ (resp. $\mu_{y,n} = \frac{1}{n}\sum_{i=1}^n \delta_{y_i}$ and $\mu_{z,n} = \frac{1}{n}\sum_{i=1}^n \delta_{z_i}$) for $(x_i)_{1\leq i\leq n}$ (resp. $(y_i)_{1\leq i\leq n}$ and $(z_i)_{1\leq i\leq n}$), n aligned points on a geodesic, so that two consecutive points are at a distance to each other equal to ϵ (resp. $\frac{1}{2\alpha}\epsilon$ and $\frac{1}{2(1-\alpha)}\epsilon$), and where ϵ is small enough so that two consecutive points are nearest neighbours. The barycenter signatures coincide for the parameter $h = \frac{2}{n}$ since $\mathrm{d}_{\mu_{x,n},h}(x_i) = \frac{1}{\sqrt{2}}\epsilon$, $\mathrm{d}_{\mu_{y,n},h}(x_i) = \frac{1}{\sqrt{2}}\frac{1}{2\alpha}\epsilon$ and $\mathrm{d}_{\mu_{z,n},h}(x_i) = \frac{1}{\sqrt{2}}\frac{1}{2(1-\alpha)}\epsilon$ for every $1 \leq i \leq n$, so that $\bar{s}_h(\delta_{\mu_{x,n}}) = s_h(\mu_{x,n}) = \delta_{\frac{1}{\sqrt{2}}\epsilon}$ and $\bar{s}_h(\alpha\delta_{\mu_{y,n}} + (1-\alpha)\delta_{\mu_{z,n}}) = \delta_{\alpha\frac{1}{\sqrt{2}}\frac{1}{2\alpha}\epsilon + (1-\alpha)\frac{1}{\sqrt{2}}\frac{1}{2(1-\alpha)}\epsilon} = \bar{s}_h(\delta_{\mu_{x,n}})$.*

S.2.2. *Signatures computation on classical examples.* In this section, we provide examples of Riemannian manifolds for which an h_0-uniform measure μ_0 exists. Then, we compute the DTM-signature to μ_0, using (2.1) in Brécheteau (2025). This requires computation of μ_0-measures of balls.

S.2.2.1. *Basic definitions on Riemannian manifolds.* A *Riemannian manifold* $(\mathscr{M}, \mathrm{G})$ with dimension $d \in \mathbb{N}^*$ is a pair comprising a smooth (connected) d-dimensional differentiable manifold $\mathscr{M}$ and a (smooth) Riemannian metric $G = (G(x))_{x\in\mathscr{M}}$. The Riemannian metric $G(x)$ is defined on the tangent space $T_x(\mathscr{M})$ and depends (smoothly) on the point $x \in \mathscr{M}$. In local coordinates, $G(x) = (G_{i,j})_{1\leq i,j\leq d}$ is a positive-definite symmetric matrix such that if $v \in T_x(\mathscr{M})$ has coordinates $v = (v_i(x))_{1\leq i\leq d}$, then $\|v\|_x^2 = G(x)(v,v) = \sum_{i=1}^d \sum_{j=i}^d G_{i,j} v_i(x) v_j(x)$.

A Riemannian manifold is a metric space $(\mathscr{X} = \mathscr{M}, \mathrm{d})$ with a metric d called *Riemannian distance*. If $\Gamma(\mathscr{M}, x, y) = \{\gamma : [0,1] \mapsto \mathscr{M},\ \text{piecewise } \mathscr{C}^1,\ \gamma(0) = x,\ \gamma(1) = y\}$ denotes the set of rectifiable paths between x and y in $\mathscr{M}$, if $l(\gamma) = \int_{t=0}^1 \|\dot{\gamma}(t)\|_{\gamma(t)} \mathrm{d}t$ denotes the length of a path $\gamma \in \Gamma(\mathscr{M}, x, y)$ (where $\dot{\gamma}(t)$ is the tangent vector at time t), then, the geodesic distance $\mathrm{d}(x,y)$ is defined as the length of the smallest path (a geodesic path) between x and y: $\mathrm{d}(x,y) = \inf_{\gamma\in\Gamma(\mathscr{M},x,y)} l(\gamma)$.

The *injectivity radius* of $(\mathscr{M}, \mathrm{G})$ is defined as half of the smallest length of a non trivial path from a point $x \in \mathscr{M}$ to itself: $\mathscr{I}(\mathscr{M}) = \frac{1}{2}\inf_{x\in\mathscr{M}} \inf_{\gamma\in\Gamma(\mathscr{M},x,x),\gamma\neq(t\mapsto x)} l(\gamma)$. Equivalently, $\mathscr{I}(\mathscr{M})$ is the maximal value r so that, for every point in $\mathscr{M}$, the geodesic ball with radius r has no double point.

The *Riemannian measure* is defined in local coordinates as the Borel measure with density with respect to the Lebesgue measure given by $\sqrt{\det(G(x))}$. For compact manifolds, we renormalise this density so that μ_0 is a probability measure. If the Riemannian measure is always quasi-uniform, in the sense of Definition S.1.1 Christensen (1970), for special examples of Riemannian manifolds, the Riemanian measure is uniform. This is the case of connected *homogeneous Riemannian manifolds* Chavel (2006), that is connected Riemannian manifolds with the property that the group of isometries of $(\mathscr{M}, \mathrm{G})$ acts transitively on $\mathscr{M}$, that is, if to each $x, y \in \mathscr{M}$, there exists an isometry ϕ of $(\mathscr{M}, \mathrm{G})$, such that $\phi(x) = y$. By isometry of $(\mathscr{M}, \mathrm{G})$, we mean a diffeomorphism ϕ that is isometric in the sense that $G(x)(v,w) = G(\phi(x))(\phi_*(v), \phi_*(w))$, where $\phi_* : T\mathscr{M} \mapsto T\mathscr{M}$ denotes the induced bundle map (in local coordinates, the Jacobian linear transformation) linear on each fiber. Chavel (2006) Such an isometry is also an isometry for the geodesic distance d and preserves volumes of balls.

S.2.2.2. *Examples of computations of sampling and barycenter signatures.* In this section, we first display the sampling (true) signatures together with the barycenter signatures based on Monte-Carlo simulations, for the unit circle $\mathbb{S}^1$, the sphere $\mathbb{S}^2$, the flat torus $\mathbb{T}^2$ and the Bolza surface $\mathbb{B}$. Then, we provide the values of the true signatures for these four examples.

Quantiles of signatures of μ_0, $s_h(\mu_0)$ and of barycenters of signatures, $\bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$, for $n \in \{20, 50, 100, 1000\}$ obtained by Monte-Carlo approximation with 10000 samples (100 for the Bolza surface), are available in Figure S.1. This figure illustrates the convergence of the barycenter of signatures to the signature of μ_0 when n goes to ∞, as noticed in Proposition 2.7 in Brécheteau (2025).

Quantiles of barycenter signatures $\bar{s}_h(\hat{\mathbb{\mu}}_{0,n})$ and of signatures based on 20 n-samples on the circle, $\bar{s}_h(\hat{\boldsymbol{\mu}}_{0,n})$, for $n \in \{20, 50, 100, 1000\}$, are available in Figure S.2. This figure illustrates how the variance of the quantile signatures around their barycenter decreases when the sample size n increases, and when the parameter h decreases.

S.2.2.2.1. The circle $\mathbb{S}^1$. The circle

$$\begin{aligned}\mathbb{S}^1_R &= \{p_\theta := (R\cos\theta, R\sin\theta),\, \theta\in\mathbb{R}\}\\ &\simeq \mathbb{R}/\mathbb{Z},\end{aligned}$$

with radius $R > 0$, equipped with the metric $G(p_\theta)(v,w) = R^2 vw$ for every $\theta \in \mathbb{R}$ and $v, w \in \mathbb{R}$, is an homogeneous Riemannian manifold. For every $\theta_1, \theta_2 \in \mathbb{R}$, an isometry ϕ of $\mathbb{S}^1_R$ so that $\phi(p_{\theta_1}) = p_{\theta_2}$ is given by $\phi(p_\theta) \mapsto p_{\theta+\theta_2-\theta_1}$. Its geodesic distance is

$$\mathrm{d}(p_{\theta_1}, p_{\theta_2}) = R\min(|\theta_1-\theta_2|, |\theta_1+1-\theta_2|, |\theta_1-\theta_2-1|), \tag{S.2.9}$$

for $\theta_1, \theta_2 \in [0, 2\pi]$[1]. Moreover, its Riemannian probability measure is

$$\mu_0(B) = \frac{1}{2\pi} Leb(\{\theta\in[0,2\pi],\, p_\theta\in B\})$$

[1]We consider the three paths $t \mapsto p_{t(\theta_2-\theta_1)+\theta_1}$, $t \mapsto p_{t(\theta_2-\theta_1-1)+\theta_1+1}$ and $t \mapsto p_{t(\theta_2-\theta_1+1)+\theta_1-1}$.

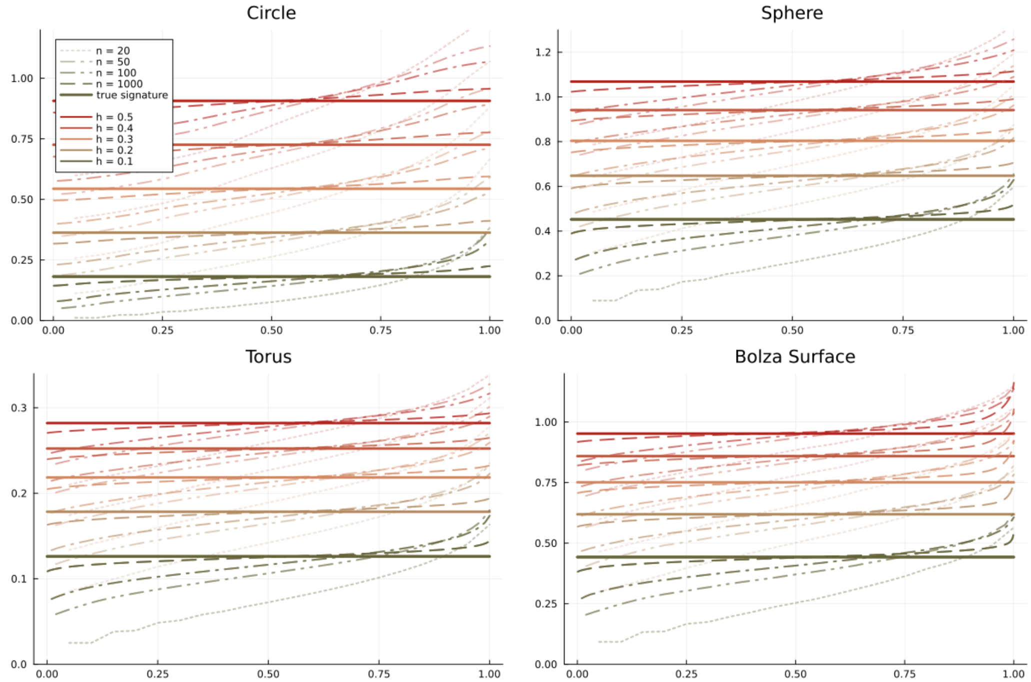

FIG S.1. *Signatures for μ_0 and barycenter signatures*

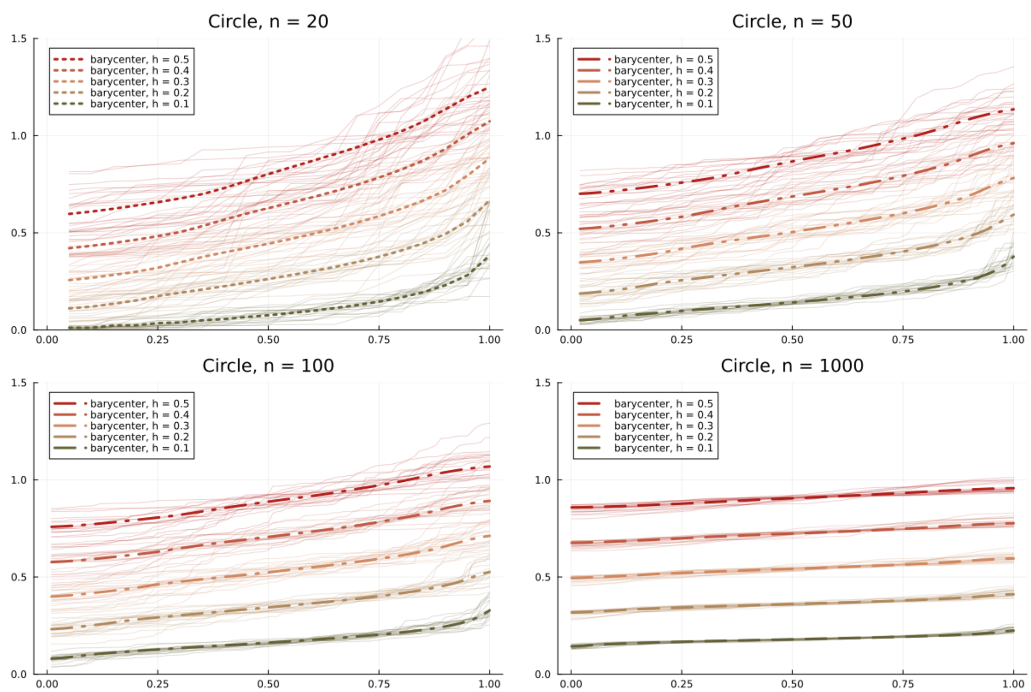

FIG S.2. *Barycenter signatures for the circle*

for every Borel set B of $\mathbb{S}^1_R$, where Leb denotes the Lebesgue measure on $\mathbb{R}$.

PROPOSITION S.2.5. *For every $h \in [0,1]$, the radius of a ball with μ_0-measure h is $\delta_{\mu_0,h} = \pi Rh$. Moreover, $s_h(\mu_0) = \delta_{\mathrm{d}_h}$ with $\mathrm{d}_h = \sqrt{\frac{1}{3}}\pi Rh$.*

The proof of Proposition S.2.5 is available in Appendix S.6.2.5.

S.2.2.2.2. The sphere $\mathbb{S}^2$. The sphere

$$\begin{aligned}\mathbb{S}^2 &= \{(x,y,z) \in \mathbb{R}^3,\, x^2+y^2+z^2 = 1\}\\ &= \{p_{\theta,\phi} := (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi),\, (\theta,\phi) \in \mathbb{R}^2\},\end{aligned}$$

in spherical coordinates, equipped with the metric $G(p_{\theta,\phi})(v,w) = v_\theta w_\theta + \sin^2(\theta) v_\phi w_\phi$ for every $(\phi,\theta) \in \mathbb{R}^2$ and $v = (v_\theta, v_\phi), w = (w_\theta, w_\phi) \in \mathbb{R}^2$, is an homogeneous Riemannian manifold. Isometries are given by rotations, in $SO_3(\mathbb{R})$, that act transitively on $\mathbb{S}^2$. The geodesic distance is

$$\mathrm{d}(p_{\theta_1,\phi_1}, p_{\theta_2,\phi_2}) = \arccos(\langle p_{\theta_1,\phi_1}, p_{\theta_2,\phi_2}\rangle), \tag{S.2.10}$$

that is,

$$\mathrm{d}(p_{\theta_1,\phi_1}, p_{\theta_2,\phi_2}) = \arccos\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\right) \tag{S.2.11}$$

for every pairs $(\theta_1,\phi_1),(\theta_2,\phi_2) \in \mathbb{R}^2$.[2] The Riemannian probability measure is

$$\mu_0(B) = \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathbb{1}_B(p_{\theta,\phi})\sin\theta \mathrm{d}\theta\mathrm{d}\phi \tag{S.2.12}$$

for every Borel set B of $\mathbb{S}^2$.

PROPOSITION S.2.6. *For every $h \in (0,1]$, the radius of a ball with μ_0-measure h is $r_h := \delta_{\mu_0,h} = \arccos(1-2h)$ if $h \le \frac{1}{2}$, and $r_h := \delta_{\mu_0,h} = \pi - \arccos(2h-1)$ if $h \ge \frac{1}{2}$. Moreover, $s_h(\mu_0) = \delta_{\mathrm{d}_h}$ with*

$$\mathrm{d}_h = \sqrt{\frac{1}{2h}\left(-r_h^2\cos(r_h) + 2r_h\sin(r_h) + 2\cos(r_h) - 2\right)}. \tag{S.2.13}$$

The proof of Proposition S.2.6 is available in Appendix S.6.2.6.

S.2.2.2.3. The flat torus $\mathbb{T}^2$. The flat torus

$$\begin{aligned}\mathbb{T}^2 &= \mathbb{R}^2/\mathbb{Z}^2\\ &= \{p = (x \mod 1, y \mod 1),\, (x,y) \in \mathbb{R}^2\},\end{aligned}$$

equipped with the metric $G(p)(v,w) = v_x w_x + v_y w_y$ for every $p \in \mathbb{T}^2$ and $v = (v_x, v_y), w = (w_x, w_y) \in \mathbb{R}^2$, is an homogeneous Riemannian manifold. For every $p_1, p_2 \in \mathbb{T}^2$, an isometry that sends p_1 to p_2 is given by $\phi : p \in \mathbb{T}^2 \mapsto p - p_1 + p_2$. The geodesic distance is given by

$$\begin{aligned}\mathrm{d}^2(p_1,p_2) = &\min(|x_1-x_2|, |x_1-x_2-1|, |x_1-x_2+1|)^2\\ &+ \min(|y_1-y_2|, |y_1-y_2-1|, |y_1-y_2+1|)^2\end{aligned}$$

[2]For instance, for $\theta = \frac{\pi}{2}$, the geodesic distance is obtained with the path $(\gamma_t)_{0\le t\le 1}$ given by $\gamma_t = p_{\frac{\pi}{2}, t(\phi_2-\phi_1)+\phi_1}$. Then, $\|\dot\gamma_t\|_{\gamma_t} = 0 + \sin\frac{\pi}{2}|\phi_2-\phi_1| = |\phi_2-\phi_1|$, thus, $\mathrm{d}(p_{\frac{\pi}{2},\phi_1}, p_{\frac{\pi}{2},\phi_2}) = |\phi_2-\phi_1| = \arccos\left(\cos^2\frac{\pi}{2} + \sin^2\frac{\pi}{2}\cos(\phi_1-\phi_2)\right)$.

for $p_1, p_2 \in \mathbb{T}^2$. The Riemannian probability measure is

$$\mu_0(B) = Leb(\{(x,y) \in [0,1]^2,\ (x,y) \in B\}), \tag{S.2.14}$$

where Leb is the Lebesgue measure on $\mathbb{R}^2$, for every Borel set B of $\mathbb{T}^2$.

The injectivity radius of the flat Torus $\mathbb{T}^2$ is given by $\mathscr{I}(\mathbb{T}^2) = 0.5$, since the geodesic $\gamma_t = (t, 0)$, for $t \in [0,1]$ is of length 1, with $\gamma_0 = \gamma_1 = (0,0)$, and no geodesic curve has a smaller length.

PROPOSITION S.2.7. *For every $h \leq \frac{\pi}{4} \simeq 0.7854$ so that $\sqrt{\frac{h}{\pi}} \leq \mathscr{I}(\mathbb{T}^2) = 0.5$, the radius of a ball with μ_0-measure h is $\delta_{\mu_0,h} = \sqrt{\frac{h}{\pi}}$. Moreover, $s_h(\mu_0) = \delta_{\mathrm{d}_h}$ with*

$$\mathrm{d}_h = \sqrt{\frac{h}{2\pi}}. \tag{S.2.15}$$

The proof of Proposition S.2.7 is available in Appendix S.6.2.7.

S.2.2.2.4. The Bolza surface $\mathbb{B}$ in the Poincaré disk. The Poincaré disk is

$$\mathbb{D}^2 = \{x + iy \in \mathbb{C},\ x, y \in \mathbb{R}, x^2 + y^2 < 1\},$$

equipped with the Riemannian metric $G(x+iy)(v,w) = 4\frac{v_x w_x + v_y w_y}{(1-x^2-y^2)^2}$ for every $x + iy \in \mathbb{D}^2$ and $v = (v_x, v_y), w = (w_x, w_y) \in \mathbb{R}^2$ (Bonahon, 2009, Section 2.7), is an homogeneous Riemannian manifold. Isometries of $\mathbb{D}^2$ are given by $\phi : p \in \mathbb{D}^2 \mapsto \frac{\alpha p + \beta}{\bar{\beta} p + \bar{\alpha}}$ with $\alpha, \beta \in \mathbb{C}$ so that $|\alpha|^2 - |\beta|^2 = 1$ (Bonahon, 2009, Proposition 2.23), with inverse given by $\phi^{-1} : q \in \mathbb{D}^2 \mapsto \frac{-q\bar{\alpha} + \beta}{\bar{\beta} q - \alpha}$. For every $p \in \mathbb{D}^2$, setting $\alpha = \frac{1}{\sqrt{1-|p|^2}}$ and $\beta = -\alpha p$ provides an isometry that sends p to 0. The geodesic distance is

$$\mathrm{d}(p_1, p_2) = \operatorname{argch}\left(1 + \frac{2\|p_1 - p_2\|^2}{(1 - \|p_1\|^2)(1 - \|p_2\|^2)}\right) \tag{S.2.16}$$

for $p_1, p_2 \in \mathbb{D}^2$[3]. The Riemannian measure is $\tilde{\mu}_0$ given by

$$\tilde{\mu}_0(B) = \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} \mathbb{1}_B((x,y)) \mathbb{1}_{x^2+y^2<1} \frac{4}{(1 - x^2 - y^2)^2} \mathrm{d}x\mathrm{d}y \tag{S.2.17}$$

and equivalently, by

$$\tilde{\mu}_0(B) = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \mathbb{1}_B((r\cos(\theta), r\sin(\theta))) \frac{4r}{(1 - r^2)^2} \mathrm{d}r\mathrm{d}\theta, \tag{S.2.18}$$

for every Borel set B of $\mathbb{D}^2$. The measure $\tilde{\mu}_0$ is not a probability measure since $\mathbb{D}^2$ is not compact.

In order to get a compact Riemannian manifold from $\mathbb{D}^2$, we identify points in $\mathbb{D}^2$ that are equal up to a transformation of the Fuchsian group of isometries[4] of $\mathbb{D}^2$ which generator elements $(f_k)_{1 \leq k \leq 8}$ are given by:

$$f_k = \begin{pmatrix} 1+\sqrt{2} & (2+\sqrt{2})\sqrt{\sqrt{2}-1}\, e^{\frac{ik\pi}{4}} \\ (2+\sqrt{2})\sqrt{\sqrt{2}-1}\, e^{-\frac{ik\pi}{4}} & 1+\sqrt{2} \end{pmatrix}. \tag{S.2.19}$$

[3] In particular, if $p_1 = t_1 v$ and $p_2 = t_2 v$ are aligned along some vector $v \in \mathbb{D}^{2*}$ and $0 \leq t_1 \leq t_2 \leq 1$, then, $\mathrm{d}(p_1, p_2) = 2(\operatorname{argth}(t_2) - \operatorname{argth}(t_1))$.

[4] The maps $z \mapsto f_k(z) = \frac{\alpha_k z + \beta_k}{\bar{\beta}_k z + \bar{\alpha}_k}$ with $\alpha_k = (f_k)_{1,1}$ and $\beta_k = (f_k)_{1,2}$ are isometries of $\mathbb{D}^2$, since $|\alpha_k|^2 - |\beta_k|^2 = 1$.

The Bolza surface, $\mathbb{B}$, is a fundamental domain of the action of the Fuchsian group on $\mathbb{D}^2$, given by the octagon delimited by the boundaries of the eight disks $(D_k)_{1\leq k\leq 8}$ given by:

$$D_8 = \mathrm{B}_{x_0,r_0}, \text{ with } x_0 = \frac{1+\frac{1}{\sqrt{2}}}{\frac{2}{\sqrt{\sqrt{2}}}\cos\left(\frac{\pi}{8}\right)} \text{ and } r_0 = \sqrt{x_0^2-1}, \tag{S.2.20}$$

and $D_k = e^{2i\pi\frac{k}{8}}D_8$ for $k \in [\![1,8]\!]$[5]. Each generator of the Fuchsian group sends an edge of the octagon to the opposite edge[6]. The *Bolza surface* is a compact homogeneous Riemannian manifold since $\mathbb{D}^2$ is an homogeneous Riemannian manifold and the Fuschsian group is a group of isometries. The Riemannian probability measure

$$\mu_0 = \frac{1}{\tilde{V}}\tilde{\mu}_0 \tag{S.2.21}$$

on $\mathbb{B}$, with the $\tilde{\mu}_0$-measure of the Bolza surface given by $\mathscr{V} = \tilde{\mu}_0(\mathbb{B}) \simeq 12.5664$, according to Lemma S.6.2. Its injectivity radius $\mathscr{I}(\mathbb{B}) \leq 2\,\mathrm{argth}(x_0 - r_0) \simeq 1.1437$, the geodesic distance between 0 and $x_0 - r_0$. [7].

The Poincaré disk and the Bolza surface are also quotients of the group $SL_2(\mathbb{R})$ Ratner (1987); Faure (2023). This specificity is used in Section S.4.2 to compute geodesics.

PROPOSITION S.2.8. *For every h so that $\delta_{\mu_0,h} \leq \mathscr{I}(\mathrm{B})$, the radius of a geodesic ball with μ_0-measure h is $\delta_{\mu_0,h} = 2\,\mathrm{argth}(\sqrt{\frac{\mathscr{V}h}{4\pi+\mathscr{V}h}})$. Moreover, $s_h(\mu_0) = \delta_{\mathrm{d}_h}$ with*

$$\mathrm{d}_h = \sqrt{2\left(\mathrm{argsh}^2\left(\sqrt{\frac{h\mathscr{V}}{4\pi}}\right) + \left(\left(\mathrm{argsh}\left(\sqrt{\frac{h\mathscr{V}}{4\pi}}\right)\right)\sqrt{\frac{4\pi}{h\mathscr{V}}+1} - 1\right)^2\right)}. \tag{S.2.22}$$

The proof of Proposition S.2.8 is available in Appendix S.6.2.8.

S.2.3. *Median signatures.* For the family of tests $(\phi^{\mathrm{iid}}_{n,h})_{h\in(0,1)}$, we decided to use the Wasserstein barycenters, that minimise the expectation of the squared L_2-Wasserstein distance. We may have used the Wasserstein medians, that minimise the expectation of the L_1-Wasserstein distance. However, the stability of Wasserstein medians would be in L_∞ Wasserstein (Carlier, Chenchene and Eichinger, 2023, Section 3.1), which is not satisfactory.

We do not expect a better upper bound than the one given by Proposition 2.6 in Brécheteau (2025), based on the stability result of (Carlier, Delalande and Mérigot, 2024, Section 1.2.2), that is, $\mathcal{W}_2(\bar{\mathbb{µ}}_1,\bar{\mathbb{µ}}_2) \leq \mathcal{W}_1(\mathbb{µ}_1,\mathbb{µ}_2)$. For instance, for a measure $\mathbb{µ}_1 = \frac{1}{3}\delta_{\delta_0} + \frac{1}{3}\delta_\mu + \frac{1}{3}\delta_{\delta_1}$ with μ supported on $[0,1]$ and $\mathbb{µ}_2 = \frac{1}{3}\delta_{\delta_0} + \frac{1}{3}\delta_\nu + \frac{1}{3}\delta_{\delta_1}$, then, $\mathcal{W}_1(med(\mathbb{µ}_1), med(\mathbb{µ}_2)) = \mathcal{W}_1(\mu,\nu) \not\leq \mathcal{W}_1(\mathbb{µ}_1,\mathbb{µ}_2) = \frac{1}{3}\mathcal{W}_1(\mu,\nu)$.

The Wasserstein median is not unique, but existence is discussed in Carlier, Chenchene and Eichinger (2023) for the Wasserstein barycenter of a finite number of probability measures. If Wasserstein medians fail to satisfy all nice properties of Wasserstein barycenters, they may have some nice properties relatively to robustness against outliers, and are not worthless. Therefore, one may be interested in defining median signatures, as follows.

[5]The extremal points of $\mathbb{B}$ are given by $\partial D_k \cap \partial D_{k+1} \cap \mathbb{D}^2 = 2^{-\frac{1}{4}}e^{2i\pi\frac{k}{8}+i\frac{\pi}{8}}$, for $k \in [\![1,8]\!]$, with the convention $D_9 = D_1$.

[6]For every $k \in [\![1,8]\!]$, f_k sends $\partial D_{k+4} \cap \partial\mathbb{B}$ to $\partial D_k \cap \partial\mathbb{B}$, where D_{k+4} is identified to D_{k-4}.

[7]According to Proposition S.2.8, the value of h corresponding to the radius $2\,\mathrm{argth}(x_0 - r_0)$ is given by $h = \frac{4\pi}{\mathscr{V}}\frac{(x_0-r_0)^2}{1-(x_0-r_0)^2} \simeq 0.7071$.

DEFINITION S.2.1. *The empirical DTM-signature median $\bar{s}_{n,h,1}(\hat{\mathbb{\mu}}_{0,n})$ to the uniform distribution is defined as the Wasserstein median of the distribution $\mathbb{s}_{n,h}(\hat{\mathbb{\mu}}_{0,n})$ of $s_h(\hat{\boldsymbol{\mu}}_{0,n})$, where $\hat{\boldsymbol{\mu}}_{0,n} \sim \hat{\mathbb{\mu}}_{0,n}$ on $\mathscr{P}([0,\mathscr{D}(\mathscr{X})])$.*

$$\bar{s}_{n,h,1}(\mathbb{\mu}_{0,n}) \in \arg \min_{s \in \mathscr{P}([0,\mathscr{D}(\mathscr{X})])} \mathbb{E}_{\boldsymbol{s} \sim \mathbb{s}_{n,h}(\hat{\mathbb{\mu}}_{0,n})} \left[\mathcal{W}_1(s, \boldsymbol{s})\right]. \tag{S.2.23}$$

In practice, median empirical DTM-signatures are approximated by a Monte Carlo procedure. Moreover, the median $\bar{s}_{n,h,1}(\hat{\mathbb{\mu}}_n)$ converges to $s_h(\mu)$, uniformly on $\mu \in \mathscr{P}(\mathscr{X})$:

PROPOSITION S.2.9.

$$\sup_{\mu \in \mathscr{P}(\mathscr{X})} \mathcal{W}_1(s_h(\mu), \bar{s}_{n,h,1}(\hat{\mathbb{\mu}}_n)) \to 0,\, n \to \infty. \tag{S.2.24}$$

The proof of Proposition S.2.9 is available in Section S.6.2.9.

S.3. Numerical illustrations - Supplement.

S.3.1. *Supplement to Section 4.1 in Brécheteau (2025).*

S.3.1.1. *Definition of test aggregation and multiple testing procedures.*

S.3.1.1.1. Aggregation of tests. We implement the aggregation method of Fromont and Laurent (2006), that work as follows. For a collection of positive weights $(\omega_i)_{i\in I}$ satisfying $\sum_{i\in I} \omega_i = 1$, we define the aggregated test $\boldsymbol{\phi}_{n,\alpha}$ as

$$\boldsymbol{\phi}_{n,\alpha} = 1 \Leftrightarrow \sup_{i\in I} \mathbf{T}_{n,h_i} - \mathrm{q}_{1-\mathrm{u}_\alpha \omega_\mathrm{i},\mathrm{n},\mathrm{h}_\mathrm{i}} > 0, \tag{S.3.1}$$

meaning that H_0 is rejected when $\mathbf{T}_{n,h_i} > \mathrm{q}_{1-\mathrm{u}_\alpha \omega_\mathrm{i},\mathrm{n},\mathrm{h}_\mathrm{i}}$ for some $i \in I$, where

$$u_\alpha = \sup\left\{ u > 0,\, \mathbb{P}_{\hat{\mathbb{\mu}}_{0,n}}\left(\sup_{i\in I} \mathbf{T}_{n,h_i} - \mathrm{q}_{1-\mathrm{u}\omega_\mathrm{i},\mathrm{n},\mathrm{h}_\mathrm{i}} > 0\right) \le \alpha \right\}, \tag{S.3.2}$$

and where $\mathrm{q}_{1-\mathrm{u}_\alpha \omega_\mathrm{i},\mathrm{n},\mathrm{h}_\mathrm{i}}$ is the $1-u_\alpha\omega_i$-quantile of the distribution of $\mathbf{T}_{n,h_i}$, under H_0 (that is, when $\mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}$). Since the null hypothesis is simple, the value of u_α can be estimated via a Monte-Carlo procedure, so that the test has a type I error of α. As for individual tests, the rejection of the null hypothesis occurs if at least one of the statistics is too large.

S.3.1.1.2. Benjamini Hochberg procedure. The Benjamini-Hochberg procedure consists in reordering the p-values $(\mathbf{p}_{n,h_i})_{i\in I}$ so that $\mathbf{p}_{n,h_{(1)}} \le \mathbf{p}_{n,h_{(2)}} \le \ldots \le \mathbf{p}_{n,h_{(n)}}$. For $\alpha \in (0,1)$, let $\tilde{\mathscr{R}}_\alpha = \{i \in I,\, \mathbf{p}_{n,h_{(i)}} \le \frac{\alpha i}{|I|}\}$. Let $i_0 = \max(\tilde{\mathscr{R}}_\alpha)$. Then, $\mathscr{R}_\alpha = \{i \in I,\, \mathbf{p}_{n,h_i} \le \mathbf{p}_{n,h_{(i_0)}}\}$ denotes the indices of tests for which H_0 is rejected. Then, based on the Benjamini-Hochberg procedure, the hypothesis $\mathrm{H}_0 : \mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}$ is rejected when $\mathscr{R}_\alpha \neq \emptyset$. This procedure is not proved to be of level α, since the independence of the test statistics is not satisfied, Benjamini and Yekutieli (2001). However, we show numerically that this testing procedure has level α on the considered examples.

S.3.1.1.3. Bonferroni procedure. In the Bonferroni procedure, the i-th test rejects H_0 when $\mathbf{p}_{n,h_i} \le \frac{\alpha}{|I|}$. Let $\mathscr{R}_\alpha = \{i \in I,\, \mathbf{p}_{n,h_i} \le \frac{\alpha}{|I|}\}$ denotes the indices of tests for which the null hypothesis H_0 is rejected. Then, based on the Bonferroni procedure, the hypothesis $\mathrm{H}_0 : \mathbb{\mu}_n = \hat{\mathbb{\mu}}_{0,n}$ is rejected when $\mathscr{R}_\alpha \neq \emptyset$. This test is of level α.

S.3.1.2. *Selection of the scale parameter.* In this part, associated with Section 4.1, we display the results of the same experiments done for $\mathbb{S}^1$ in Section 4.1 in Brécheteau (2025), for the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$ and the Bolza surface $\mathbb{B}$.

We consider the following sampling procedures:

- On $\mathbb{S}^2$: the von Mises-Fisher distribution with parameter $\kappa = 0.5$; a mixture of 6 von Mises-Fisher distributions with centers on a regular polytope, with parameter $\kappa = 10$
- On $\mathbb{T}^2$: the Normal distribution with standard deviation 0.1 ; a mixture of 4 Normal distributions with standard deviation 0.04, with centers $(0, 0.5)$, $(0, -0.5)$, $(0.5, 0)$, $(-0.5, 0)$
- On $\mathbb{B}$: a distribution based on 10 iterations of a Brownian motion, with center 0, with standard deviation 6 ; a mixture of 4 distributions based on 10 iterations of a Brownian motion with standard deviation 1.5, with centers $(0, -0.5)$, $(0, 0.5)$, $(-0.5, 0)$, $(0.5, 0)$.

In Figures S.3, S.4, and S.5, 100-samples of the different procedures available.

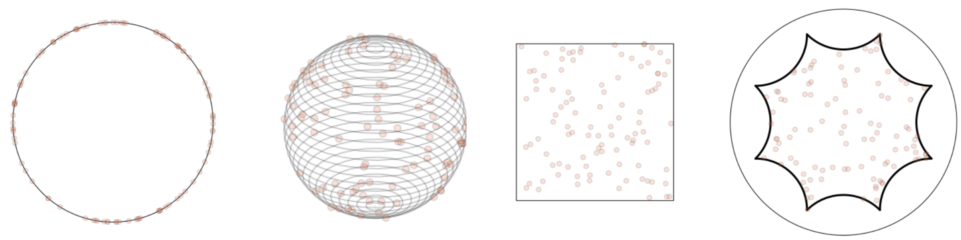

FIG S.3. *Samples of uniform distributions, used in Section 4.1 in Brécheteau (2025).*

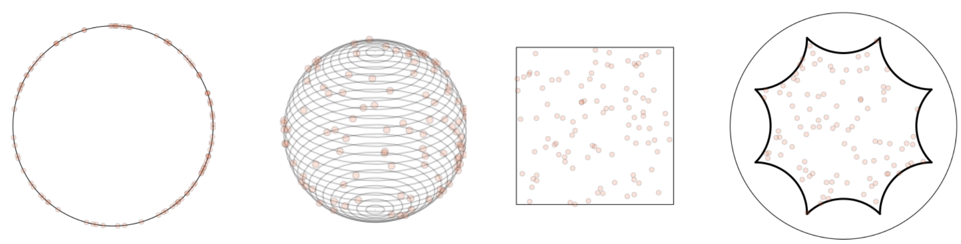

FIG S.4. *Samples of von Mises-Fisher or Normal distributions, used in Section 4.1 in Brécheteau (2025).*

In Table 1, the power approximations of the tests are computed, and in Figure S.6, the proportion of parameters h selected are displayed.

S.3.2. *Supplement to Section 4.2 in Brécheteau (2025) Comparison to classical tests of uniformity.* In this part, associated with Section 4.2 in Brécheteau (2025), we represent the p-values of the different tests of uniformity for the grid on the circle $\mathbb{S}^1$ and for the Sobol sequence on the sphere $\mathbb{S}^2$, in Figure S.7. Then, we represent the power for the differents tests of nominal level 1% and 10% on $\mathbb{S}^1$ in Figure S.8 and on $\mathbb{S}^2$ in Figure S.9. These two figures strenghten the results obtained in Figure 3 in Brécheteau (2025) for the nominal level 5%.

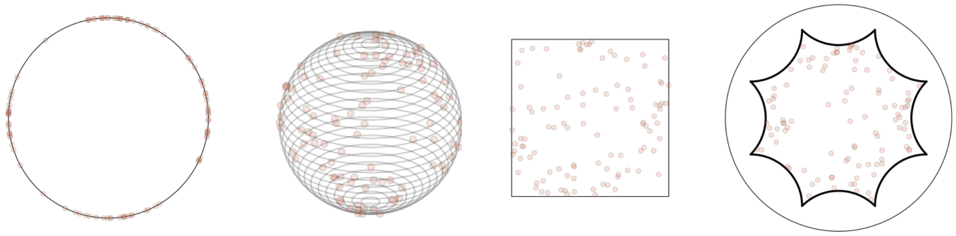

FIG S.5. *Samples of mixtures of von Mises-Fisher or Normal distributions, of Section 4.1 in Brécheteau (2025).*

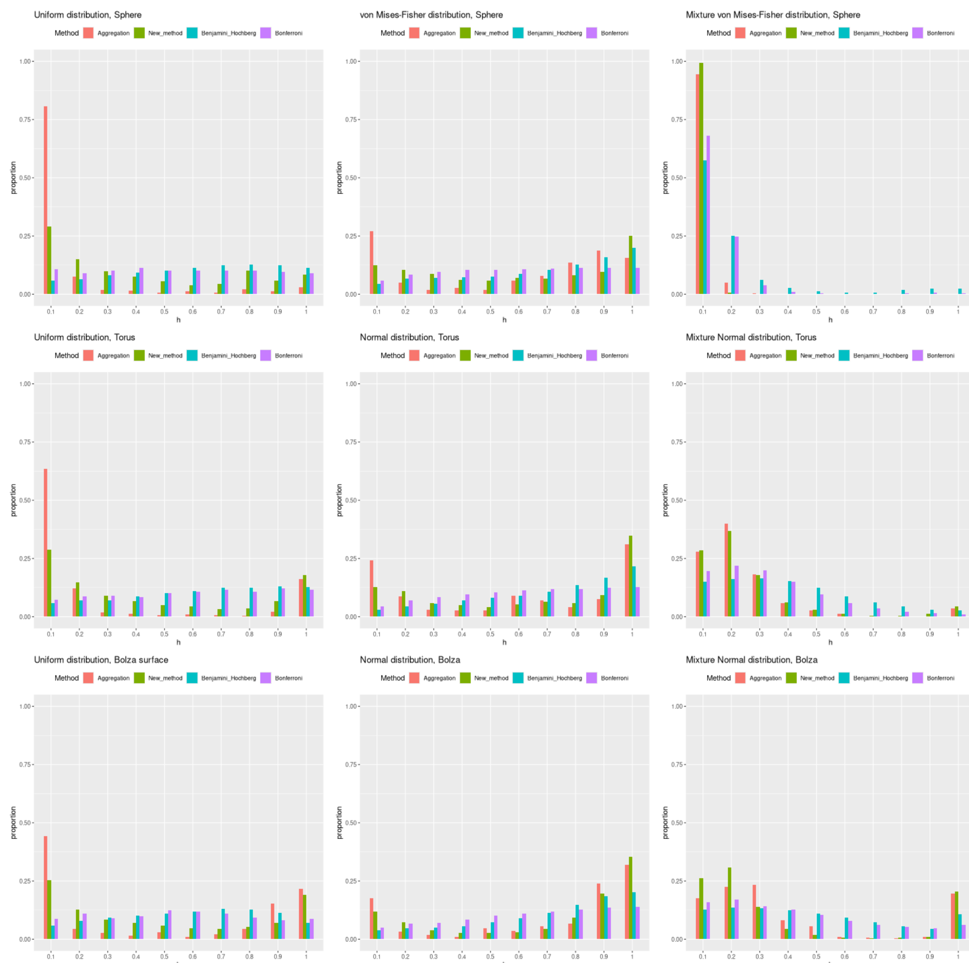

FIG S.6. *Selection of parameters h.*

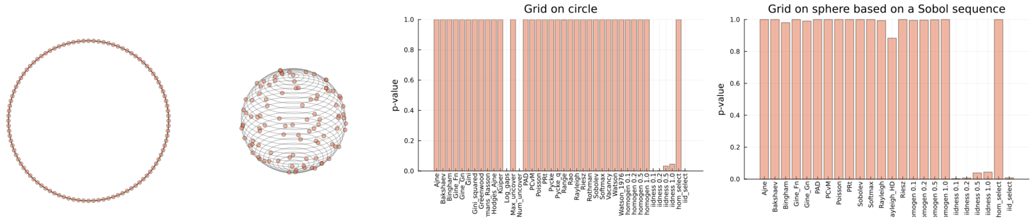

FIG S.7. *Regular samples on $\mathbb{S}^1$ and $\mathbb{S}^2$ (left), p-values for the tests on these samples (right)*

Space	Method	Class. Aggregation	Alt. Aggregation	Benj.-Hoch.	Bonferroni
$\mathbb{S}^2$	Uniform	0.0496	0.0514	0.0364	0.0146
$\mathbb{S}^2$	von Mises-Fisher	0.5802	0.5429	0.5689	0.3824
$\mathbb{S}^2$	Mixture von Mises-Fisher	0.9785	0.9792	0.9434	0.943
$\mathbb{T}^2$	Uniform	0.0481	0.0544	0.0376	0.0129
$\mathbb{T}^2$	Normal	0.4876	0.5024	0.4739	0.3105
$\mathbb{T}^2$	Mixture Normal	0.5972	0.6171	0.4569	0.3734
$\mathbb{B}$	Uniform	0.0476	0.0479	0.0344	0.0141
$\mathbb{B}$	Brownian	0.4048	0.4098	0.3562	0.2356
$\mathbb{B}$	Mixture Brownian	0.7604	0.7898	0.6879	0.5672

TABLE 1

Power comparison for tests of level 0.05

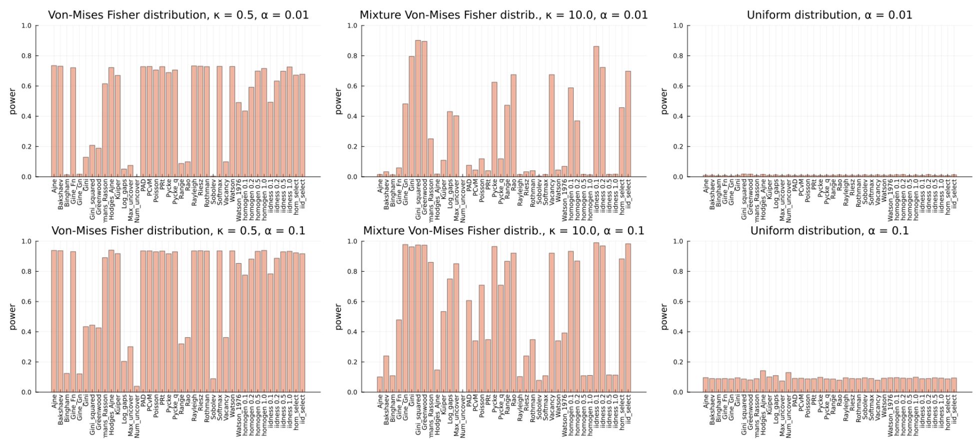

FIG S.8. *Powers on the circle*

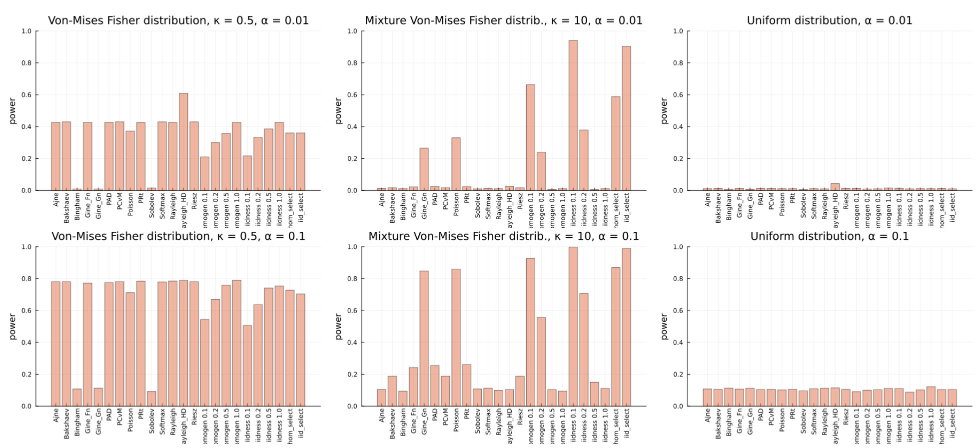

FIG S.9. *Powers on the sphere*

S.3.3. *Supplement to Section S.3.3 in Brécheteau (2025): Application to shape analysis.* In this section, we consider sets of points on several spaces (the olympic rings (1), the infinity symbol (2), the sphere (3), the bunny (4), the klein bottle (5) and the projective plan (6)), with cardinality 40000 (except for the sphere, 34686 points, and the bunny, 34835 points). The spaces are 1-dimensional in $\mathbb{R}^2$ (1,2), 2-dimensional in $\mathbb{R}^3$ (3,4), 2-dimensional in $\mathbb{R}^4$

(5,6). The samples (1,2,3,4) are represented in Figure S.10, together with the normal vectors for examples (3,4).

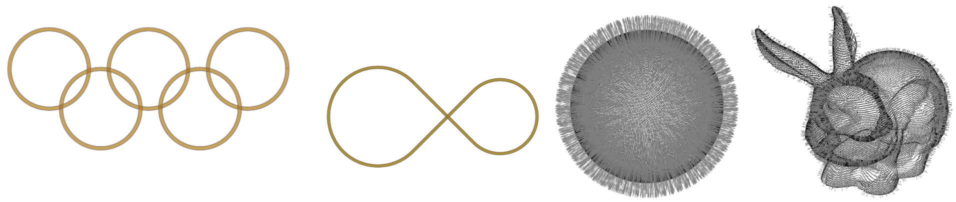

FIG S.10. *Samples : olympic rings, infinity symbol, sphere, Stanford bunny*

The olympic rings (1) are union of five circles with radii 1 and with respective centers $(2.5,0)$,$(1.3,-1)$, $(0,0)$, $(5,0)$ and $(3.8,-1)$. The infinity symbol (2) is made of two circles with centers $(0,0)$ and $(14/4,0)$, with radii $\sqrt{2}$ and $\sqrt{\frac{9}{8}}$, joined continuously by segments. The Stanford bunny (4), from the Stanford University Computer Graphics Laboratory, http://graphics.stanford.edu/data/3Dscanrep/ and the regular grid on the sphere (3) have been generated from the python notebook Buet, Leonardi and Masnou, (for the sphere, with `generateEllipsoid(1, 1, 331)`). The Klein bottle (5) and the projective plan (6) have been generated according to the procedure described in Bobrowski and Skraba (2022). For the Klein bottle, given two random variables θ and ϕ on $[0,2\pi]$, we define the four coordinates of a random vector by:

- $X_1 = (1+\cos\theta)\cos\phi$,
- $X_2 = (1+\cos\theta)\sin\phi$,
- $X_3 = \sin\theta\cos\frac{\phi}{2}$,
- $X_4 = \sin\theta\sin\frac{\phi}{2}$.

For the projective plan, given a random vector (U_1,U_2,U_3), uniformly distributed on the sphere $\mathbb{S}^2$, we define the four coordinates of a random vector by:

- $X_1 = U_1U_2$,
- $X_2 = U_1U_3$,
- $X_3 = U_2^2 - U_3^2$,
- $X_4 = 2U_2U_3$.

To all points, we associate an estimator of the normal vector (1,2,3,4), or the tangent space (3,4,5,6), using Buet and Rumpf (2022); Buet, Leonardi and Masnou (2022) and its Python implementation, available in Buet, Leonardi and Masnou. Regarding the normal vector, we consider both the normal vector provided by the algorithm in Buet, Leonardi and Masnou and its opposite vector. Therefore, the samples of normal vectors (in the circle (1,2) or in the sphere (3,4)) have a cardinality that is twice the sample size, and they have a rotational symmetry with respect to the origin. The tangent spaces are elements of the Grassmannian $\mathbb{G}(2,3)$ for examples (3,4) and $\mathbb{G}(2,4)$ for examples (5,6). The distribution of the normal vectors for the samples (1,2,3,4) are represented in Figure S.11.

For the olympic rings (1) (resp. for the sphere (3)), we observe a distribution of normal vectors that seems to be uniform on the circle (1) (resp. on the sphere (3)). It makes sense, since (1) is just a union of 5 circles, so the normal vectors, if well estimated (at least far from intersection points), should behave uniformly. On the contrary, for the infinity symbol (2), the

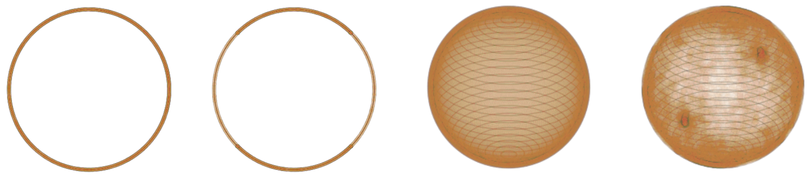

FIG S.11. *Distribution of normal vectors : olympic rings, infinity symbol, sphere, Stanford bunny*

distribution of the normal vectors is a sum of 4 dirac masses (represented by 4 points in the circle), that delimitates 4 zones with constant density given by c (left and right) and c' (top and bottom), for some constants $c < c'$. The points correspond to the normal vectors of the straight part of (2), parts with density c correspond to left and right parts of the infinity symbol, whereas parts with density c' correspond to top and bottom parts of the infinity symbol. Such directions appear more often than the previous ones. The density of the normal vectors is not uniform on the sphere. The density of the normal vectors on the bunny (4) are not uniform either, as represented in the sphere. We saw two points, that may correspond to the direction of the base of the bunny, with direction $(0, 0, 1)$ and opposite direction $(0, 0, -1)$.

We have sampled 10000 100-samples of points from the large samples, and computed the p-values for the tests of iidness, with parameter h varying in $[0.1, 0.3, 0.5, 0.7, 0.9]$. The distribution of the test statistics under the null hypothesis have been approximated by a Monte-Carlo procedure with 10000 replications, so that the p-values are computed with respect to this distribution approximation. Among the 10000 100-samples of normal vector and tangent spaces distributions, we have sorted the p-values and represented these sorted p-values in Figure S.12 and Figure S.13.

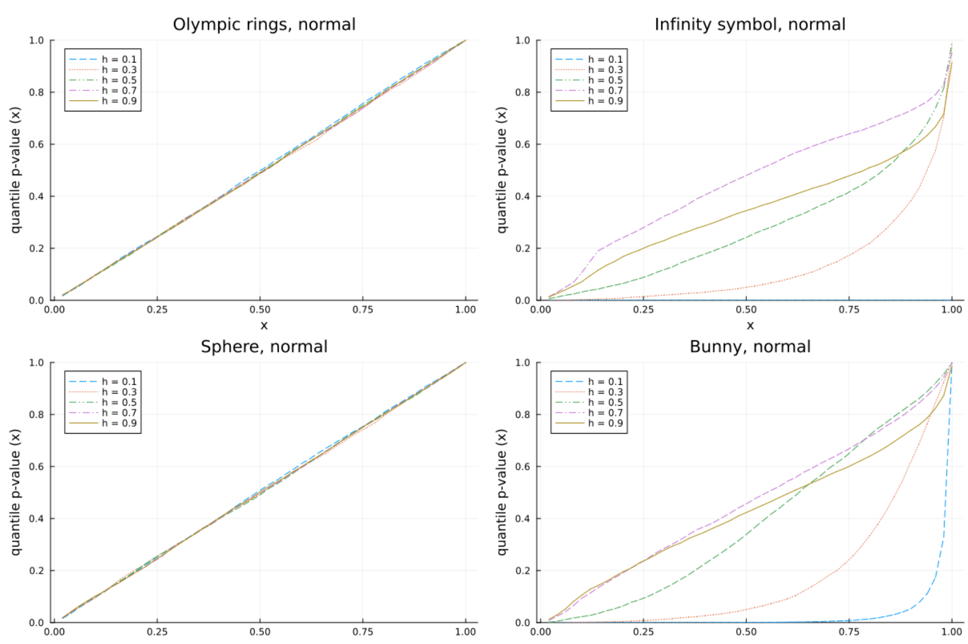

FIG S.12. *Distribution of p-values for the iidness tests, normal directions*

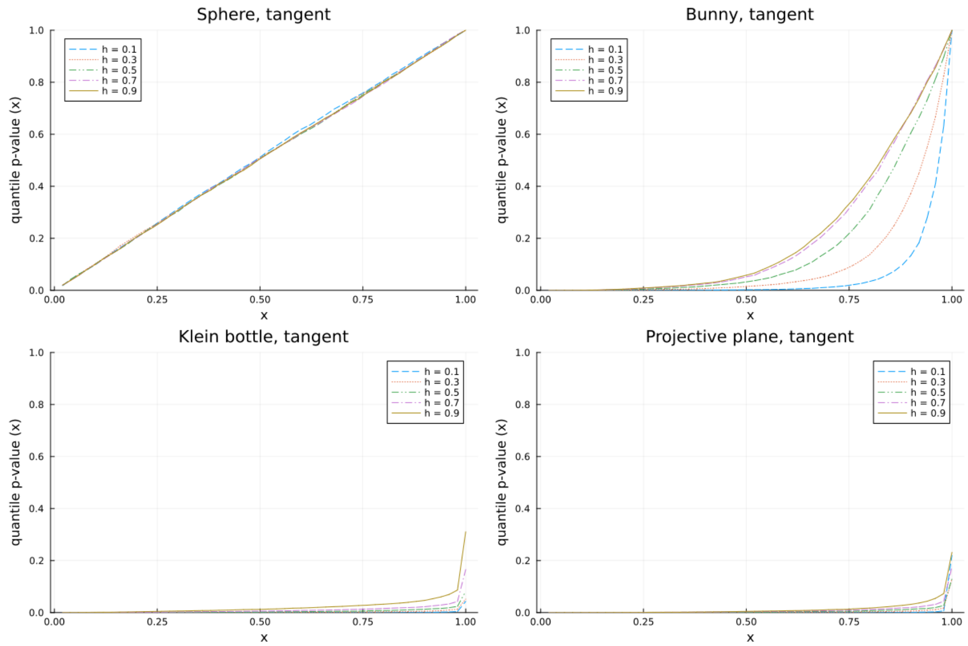

FIG S.13. *Distribution of p-values for the iidness tests, tangent planes*

Notice that these curves correspond to approximations of the quantiles of the p-values, under the alternatives. When the null hypothesis holds, we expect that the distribution of the p-values is uniform, and that the quantile function is close to the function $x \mapsto x$ on $[0,1]$, which is the case for the distribution of normal vectors of the olympic rings (1), and the normal vectors and tangent spaces for the sphere (2).

Under the alternative, we expect the quantile function to be below the function $x \mapsto x$. This is the case of all of the other examples. The rejections of the null hypothesis happen even more often when the quantile function is close to $x \mapsto 0$ on $[0,1]$, which is the case of the Klein bottle (5) and the projective plan (6). This makes sense since the dimension 2 of the tangent spaces to a surface in $\mathbb{R}^4$ is smaller than the dimension $2(4-2) = 4$ of the Grassmannian $\mathbb{G}(2,4)$. Therefore, the measure cannot be uniform. We can also notice that the small values of h provide tests that are more efficient to detect iidness and reject the most the uniformity assumption. Notice that since we sampled randomly samples of size 100, among samples of large size (approx. 40000), the samples behave as independent samples. Therefore the rejection of the iidness tests is due to the non uniformity of the underlying measure, and not on the dependance of sample points, as discussed in Section 4.3 in Brécheteau (2025).

The mean power of the tests, over the 10000 replications, for a level 0.05, are represented in the table in Table 2, and numerically confirm the above-mentioned resuts of Figure S.12 and Figure S.13.

Although such experiments may have been done by one of the numerous existing tests of uniformity on the circle, on the sphere (R package `sphunif` in García-Portugués and Verdebout (2024), with an overview of existing tests in García-Portugués and Verdebout (2018)), or on the Grassmannian Chikuse and Watson (1995); Chikuse and Jupp (2004), with an R implementation given by the function `grassmann.utest` in the `Riemann` package, You (2022), these illustrations enhance that the DTM-signature or barycenter signature for normal vector, tangent plane, or more generally, to any space generated by a subfamily of eigenvectors of local covariance matrices, provide new shape descriptors, and may be used for

	h	0.1	0.3	0.5	0.7	0.9
Olympic rings	normal	0.0489	0.0504	0.0526	0.0531	0.0527
Infinity symbol	normal	1.0	0.5027	0.1559	0.0659	0.0754
Sphere	normal	0.0531	0.0505	0.0504	0.0478	0.0475
Bunny	normal	0.896	0.4975	0.172	0.0683	0.0578
Sphere	tangent	0.0488	0.0487	0.0491	0.0499	0.0509
Bunny	tangent	0.8306	0.6821	0.5624	0.4939	0.4788
Klein bottle	tangent	1.0	0.9998	0.9989	0.9889	0.9121
Projective plane	tangent	0.9982	0.9974	0.9953	0.9868	0.9537

TABLE 2
Power of the test of iidness, based on 10000 100-samples

shape comparison, clustering, or even for testing equality of shapes, as alternatives to Mémoli (2011); Osada et al. (2002); Brécheteau (2019); Chazal et al. (2009); Chazal, De Silva and Oudot (2014).

S.4. Numerical procedures.

S.4.1. *Monte Carlo procedure to estimate the barycenter signature.* We approximate the barycenter signature $\bar{s}_h(\mathbb{P}_n)$ of a measure $\mathbb{P}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ by Monte Carlo, by $\bar{s}_h(\mathbb{P}_{n,M})$, with $\mathbb{P}_{n,M} = \frac{1}{M}\sum_{m=1}^M \delta_{\boldsymbol{\mu}_{n,M}}$ based on an M-sample $(\boldsymbol{\mu}_{n,M})_{1\leq m\leq M}$ from $\mathbb{P}_n$. Its quantile function is given by $\frac{1}{M}\sum_{m=1}^M Q_{\boldsymbol{s}_m}$, with $(\boldsymbol{s}_m = s_h(\boldsymbol{\mu}_{n,M}))_{1\leq m\leq M}$:

$$\bar{s}_h(\mathbb{P}_{n,M}) = \arg\min_{s\in\mathscr{P}([0,\mathscr{D}(\mathscr{X})])} \frac{1}{M}\sum_{m=1}^M \mathcal{W}_2^2(s, \boldsymbol{s}_m). \tag{S.4.1}$$

The estimator $\bar{s}_h(\mathbb{P}_{n,M})$ converges in distribution to $\bar{s}_h(\mathbb{P}_n)$, almost surely, when $M\to\infty$. This follows from Proposition 2.6 in Brécheteau (2025), from the fact that in compact Polish spaces $\mathcal{W}_1$ metrizes weak convergence (Villani, 2008, Theorem 6.9), and since empirical measures $\mathbb{P}_{n,M}$ converge weakly to sampling measures $\mathbb{P}_n$ almost surely Varadarajan (1958).

S.4.2. *Sampling on the Bolza surface.*

S.4.2.0.1. Sampling from the h_0-uniform measure μ_0 on the Bolza surface. Generating a random point uniformly distributed in the disk $\mathrm{B}_{0,2^{-\frac{1}{4}}}$ of the Poincaré disk is possible using two independent random variables uniform on $[0,1]$, as follows.

PROPOSITION S.4.1. *The random variable $Z = \sqrt{\frac{U}{\sqrt{2}-1+U}}\exp(2i\pi V)$, with U and V i.i.d. uniform on $[0,1]$, follows the uniform measure $\left(\tilde{\mu}_0\left(\mathrm{B}_{0,2^{-\frac{1}{4}}}\right)\right)^{-1}(\tilde{\mu}_0)_{|\mathrm{B}_{0,2^{-\frac{1}{4}}}}$.*

The proof of Proposition S.4.1 is available in Section S.6.3.1.

To generate a uniform sample on the Bolza surface, we use an acceptance-rejection method, based on a sample on the smallest ball containing the Bolza surface, $\mathrm{B}_{0,2^{-\frac{1}{4}}}$, and where points in $\bigcup_{k=1}^8 D_k$ are rejected.

S.4.3. *Generation of geodesics on the Bolza surface.* To compute geodesics, we consider the relation between the Poincaré disk $\mathbb{D}^2$ and the group $SL^2(\mathbb{R})$, Faure (2023); Ratner

(1987). Every matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ can be decomposed as a product of three matrices, accordingly to the NAK decomposition of Iwasawa[8]:

$$g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{S.4.2}$$

We can identify $SL_2(\mathbb{R})$ to $\mathbb{H}^2 \times \mathbb{S}^1$, where $\mathbb{H}^2$ is the Poincaré half space, via the map $g \mapsto (z = x + iy, \exp(i\theta))$. We can then identify $SL_2(\mathbb{R})$ to $\mathbb{D}^2 \times \mathbb{S}^1$, using the Cayley transform $z \in \mathbb{H}^2 \mapsto \omega = \frac{z-i}{z+i} \in \mathbb{D}^2$, that is an isometry[9]. Isometries of $\mathbb{H}^2$ are induced by elements $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ by, for every $(z, e^{i\theta})$:

$$z \mapsto \frac{az+b}{cz+d}, \text{ and } e^{i\theta} \mapsto \frac{c\bar{z}+d}{|c\bar{z}+d|}e^{i\theta}. \tag{S.4.3}$$

The isometry on $\mathbb{D}^2$ induced by g is $CgC^{-1} = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in SU(1,1)$ (with $|\alpha|^2 - |\beta|^2 = 1$), for $C = \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$.[10]

To compute geodesics on the Bolza surface, we compute geodesics in $\mathbb{H}^2$, apply transformations of the Fuchsian group, g_k in $\mathbb{H}^2$[11], and send points to $\mathbb{D}^2$ through the Cayley transform.

The geodesic in $\mathbb{H}$ starting at $g_0 = (z_0, \theta_0)$ with rate t is given by

$$g_t = g_0 \begin{pmatrix} \exp(t/2) & 0 \\ 0 & \exp(-t/2) \end{pmatrix},$$

so that $g_t = (z_t, \theta_t)$ with:

$$z_t = \frac{ia_0\exp(t) + b_0}{ic_0\exp(t) + d_0}, \text{ and } \exp(i\theta_t) = \frac{d_0 - ic_0\exp(t)}{|d_0 - ic_0\exp(t)|} \tag{S.4.4}$$

Consequently, using the Cayley transform, we get in $\mathbb{D}^2$ that:

$$\omega_t = \frac{i(a_0\exp(t) - d_0) + b_0 + c_0\exp(t)}{i(a_0\exp(t) + d_0) + b_0 - c_0\exp(t)}. \tag{S.4.5}$$

In particular, $d_{\mathbb{D}^2}(\omega_t, \omega_{t'}) = |t - t'|$[12]

Notice that, for a geodesic in $\mathbb{D}^2$, starting at 0, the derivative of the geodesic at time $t = 0$ is related to the initial angle θ_0: $\omega'_0 = \frac{1}{2}e^{i2\theta_0}$.

S.4.4. *Brownian motion on the Bolza surface and mixtures of Gaussian measures.* Accordingly to (Bharath et al., 2023, Section 3.1), a classical procedure to simulate a brownian motion on a compact Riemanian manifold $(\mathscr{M}, \mathrm{G})$, $(B_t)_{0\leq t\leq T}$ for $T > 0$, is to use a Markov process $(X_n^h)_{n\in[\![0,N]\!]}$, defined from a partition of $[0,T]$ with fixed time step $h = \frac{T}{N}$, $t_0 = 0 < t_1 < \ldots < t_N = T$, by:

---

[8]The relation between coefficients are: $a = y^{1/2}\cos(\theta) - xy^{-1/2}\sin(\theta)$, $b = y^{1/2}\sin(\theta) + xy^{-1/2}\cos(\theta)$, $c = -y^{-1/2}\sin(\theta)$, $d = y^{-1/2}\cos(\theta)$, $z = x + iy = \frac{ia+b}{ic+d}$, $e^{i\theta} = \frac{d-ic}{|d-ic|}$, θ.

[9]The inverse of the Cayley transform is given by $z = \phi^{-1}(\omega) = i\frac{\omega+1}{1-\omega}$.

[10]The relation between the coefficients are : $\alpha = \frac{1}{2}(a+d) + \frac{i}{2}(b-c)$, $\beta = \frac{1}{2}(a-d) + \frac{i}{2}(b+c)$, $a = Re(\alpha) + Re(\beta)$, $b = Im(\alpha) + Im(\beta)$, $c = Im(\beta) - Im(\alpha)$, $d = Re(\alpha) - Re(\beta)$.

[11]Element g_k of the Fuchsian group has coefficients: $a_k = 1 + \sqrt{2} + (2+\sqrt{2})\sqrt{\sqrt{2}-1}\cos\left(\frac{k\pi}{4}\right)$, $b_k = c_k = -(2+\sqrt{2})\sqrt{\sqrt{2}-1}\sin\left(\frac{k\pi}{4}\right)$, $d_k = 1 + \sqrt{2} - (2+\sqrt{2})\sqrt{\sqrt{2}-1}\cos\left(\frac{k\pi}{4}\right)$.

[12]This folllows from the fact that the Cayley transform is an isometry, that multiplication by g_0 in $SL_2(\mathbb{R})$ is an isometry in $\mathbb{H}$, and footnote 3.

- $X_0^h = x_0 \in \mathscr{M}$
- For every $n \in [\![0, N-1]\!]$:
  - Select a random vector $\xi_{n+1} \sim \mathscr{N}(0,h)$ on the tangent space $T_{X_n^h}(\mathscr{M})$, for the metric $\mathrm{G}(X_n^h)$;
  - Define $X_{n+1}^h = \exp_{X_n^h}(\xi_{n+1})$, the value at time $\|\xi_{n+1}\|_{\mathrm{G}(X_n^h)}$ of the geodesic starting from X_n^h with direction $\frac{\xi_{n+1}}{\|\xi_{n+1}\|_{\mathrm{G}(X_n^h)}}$.

In order to compute Brownian motion paths on the Bolza surface, we apply a slight variation of this procedure on $\mathbb{D}^2$ and apply transformations of the Fuchsian group for the paths to remain in the Bolza surface. For convenience, in $\mathbb{D}^2$, we use the symmetry of $\mathbb{D}^2$ with respect to 0: if $\phi_{n+1} : z \mapsto \frac{z+X_n^h}{\overline{X_n^h}z+1}$ is the Möbius transformation that sends the point 0 to X_n^h, then, we set

$$X_{n+1}^h = \phi_{n+1}(\exp_0(\xi_{n+1})), \tag{S.4.6}$$

with ξ_{n+1}, a random normal vector of distribution $\mathscr{N}(0,h)$ on $T_0(\mathscr{M})$. This normal vector is given by:

$$\xi_{n+1} = \frac{\sqrt{h}}{2}\sqrt{Y_{n+1}}e^{i\Theta_{n+1}}, \tag{S.4.7}$$

with Θ_{n+1} uniform on $[0,2\pi]$ and $Y_{n+1} \sim \chi^2(2)$, independent of Θ_{n+1}. Notice that $\|\xi_{n+1}\|_{\mathrm{G}(0)} = \sqrt{hY_{n+1}}$ has the same distribution as the Euclidean norm of a random vector in $\mathbb{R}^2$ with distribution $\mathscr{N}(0,\sqrt{h}I_2)$.

A Gaussian measure on the Bolza surface with center $c \in \mathbb{B}$ and variance $t > 0$ is defined as the distribution of B_t, for a Brownian path starting from $B_0 = c$. Mixtures of $k \in \mathbb{N}^*$ Gaussian measures on the Bolza surface are distributions of type $\sum_{i=1}^k \alpha_i \mathscr{N}(c_i, t_i)$, for $(\alpha_i)_{1\leq i\leq k} \in [0,1]^k$, such that $\sum_{i=1}^k \alpha_i = 1$.

S.5. Proofs of the results in the main paper, Brécheteau (2025).

S.5.1. *Proofs for Section 1 in Brécheteau (2025).*

S.5.1.1. *Proof for Theorem 1.1 in Brécheteau (2025).* According to the Prokhorov theorem, since $\mathscr{X}$ is a compact Polish space, $\mathscr{P}(\mathscr{X})$ is compact for the weak convergence. Since the weak convergence is metrized by the $\mathcal{W}_2$ distance, (Villani, 2008, Theorem 6.9), it remains to prove that the function $\mu \in \mathscr{P}(\mathscr{X}) \mapsto \mathbb{E}_\mu[\mathcal{W}_2(\hat{\boldsymbol{\mu}}_n, \mu)]$ is continuous with respect to $\mathcal{W}_2$. Let π denote the optimal transport plan between μ and ν for $\mathcal{W}_2$ (that exists according to (Villani, 2008, Theorem 4.1)) and (X_i, Y_i) an n-sample from π, then:

$$\begin{aligned}
\mathbb{E}_\mu[\mathcal{W}_2(\hat{\boldsymbol{\mu}}_n, \mu)] - \mathbb{E}_\nu[\mathcal{W}_2(\hat{\boldsymbol{\nu}}_n, \nu)] &= \mathbb{E}_\pi[\mathcal{W}_2(\hat{\boldsymbol{\mu}}_n, \mu) - \mathcal{W}_2(\hat{\boldsymbol{\nu}}_n, \nu)] \\
&\leq \mathbb{E}_\pi[\mathcal{W}_2(\hat{\boldsymbol{\mu}}_n, \hat{\boldsymbol{\nu}}_n) + \mathcal{W}_2(\mu, \nu)] \\
&\leq \mathbb{E}_\pi\left[\sqrt{\frac{1}{n}\sum_{i=1}^n \mathrm{d}^2(X_i, Y_i)}\right] + \mathcal{W}_2(\mu,\nu) \\
&\leq \sqrt{\frac{1}{n}\sum_{i=1}^n \mathbb{E}_\pi[\mathrm{d}^2(X_i,Y_i)]} + \mathcal{W}_2(\mu,\nu) \\
&= 2\mathcal{W}_2(\mu,\nu),
\end{aligned}$$

according to the Jensen inequality. The second part of the lemma is a direct consequence of the Markov inequality.

S.5.2. *Proofs for Section 2 in Brécheteau (2025).*

S.5.2.1. *Proof for Proposition 2.1 in Brécheteau (2025).* Since $\mathrm{d}_{\mu_0,h}$ is constant according to (2.5), and because of (2.6):

$$\begin{aligned}\mathcal{W}_1(s_h(\mu), s_h(\mu_0)) &= \mathcal{W}_1(\mathrm{d}_{\mu,h}{}_{\#\mu}, \mathrm{d}_{\mu_0,h}{}_{\#\mu}) \\ &\leq \int_{\mathscr{X}} |\mathrm{d}_{\mu,h}(x) - \mathrm{d}_{\mu_0,h}(x)| \,\mathrm{d}\mu(x) \\ &\leq \|\mathrm{d}_{\mu,h} - \mathrm{d}_{\mu_0,h}\|_\infty \\ &\leq \frac{1}{\sqrt{h}} \mathcal{W}_2(\mu, \mu_0).\end{aligned}$$

If π denotes the optimal transport plan between μ and ν for $W_1(\mu,\nu)$, then, according to the triangular inequality, (2.7) and (2.6):

$$\begin{aligned}\mathcal{W}_1(s_h(\mu), s_h(\nu)) &\leq \mathcal{W}_1(\mathrm{d}_{\mu,h}{}_{\#\mu}, \mathrm{d}_{\mu,h}{}_{\#\nu}) + \mathcal{W}_1(\mathrm{d}_{\mu,h}{}_{\#\nu}, \mathrm{d}_{\nu,h}{}_{\#\nu}) \\ &\leq \int_{\mathscr{X}} |\mathrm{d}_{\mu,h}(x) - \mathrm{d}_{\mu,h}(y)| \,\mathrm{d}\pi(x,y) + \|\mathrm{d}_{\mu,h} - \mathrm{d}_{\nu,h}\|_\infty \\ &\leq \int_{\mathscr{X}} \mathrm{d}(x,y)\mathrm{d}\pi(x,y) + \frac{1}{\sqrt{h}} \mathcal{W}_2(\mu,\nu) \\ &\leq \mathcal{W}_1(\mu,\nu) + \frac{1}{\sqrt{h}} \mathcal{W}_2(\mu,\nu).\end{aligned}$$

Moreover, Hölder inequality for Wasserstein distances (Villani, 2008, Remark 6.6) yields:

$$\mathcal{W}_1(s_h(\mu), s_h(\nu)) \leq \left(1 + \frac{1}{\sqrt{h}}\right) \mathcal{W}_2(\mu,\nu). \tag{S.5.1}$$

The same methods provide the bounds for the L_2-Wasserstein distance.

S.5.2.2. *Proof for Proposition 2.2 in Brécheteau (2025).* Using triangular inequality, we get that for $p = 1$ or $p = 2$:

$$\mathbb{E}_\mu \left[\mathcal{W}_p(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu))\right] \leq \mathbb{E}\left[\mathcal{W}_p\left(\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h}{}_{\#\hat{\boldsymbol{\mu}}_n}, \mathrm{d}_{\mu,h}{}_{\#\hat{\boldsymbol{\mu}}_n}\right)\right] + \mathbb{E}\left[\mathcal{W}_p\left(\mathrm{d}_{\mu,h}{}_{\#\hat{\boldsymbol{\mu}}_n}, \mathrm{d}_{\mu,h}{}_{\#\mu}\right)\right]. \tag{S.5.2}$$

Parametric rates for the left term is obtained with the following Lemma S.5.1.

Concerning the right term, after noticing that $\mathrm{d}_{\mu,h}{}_{\#\hat{\boldsymbol{\mu}}_n}$ is the empirical measure from an n-sample from $\mathrm{d}_{\mu,h}{}_{\#\mu}$, and that the support of $s_h(\mu)$ and $s_h(\hat{\boldsymbol{\mu}}_n)$ for every $\mu \in \mathscr{P}(\mathscr{X})$, $h \in (0,1]$ and $n \in \mathbb{N}$, is bounded, included in $[0, \mathscr{D}(\mathscr{X})]$, following arguments in Chazal, Massart and Michel (2016), we get that:

- Equation (2.14) is satisfied according to (Bobkov and Ledoux, 2019, Theorem 3.2).
- Equation (2.15) is satisfied according to (Bobkov and Ledoux, 2019, Theorem 7.16).
- Equation (2.17) is satisfied according to (Bobkov and Ledoux, 2019, Theorem 5.1).

- Equation (2.18) is satisfied since $\mathcal{W}_2\left(\mathrm{d}_{\mu,h_{\#\hat{\boldsymbol{\mu}}_n}}, \mathrm{d}_{\mu,h_{\#\mu}}\right) = 0$ when $\mu = \mu_0$ since $\mathrm{d}_{\mu_0,h}$ is constant.

LEMMA S.5.1. *For every $h \in (0,1]$ and $l_h > 0$, there exists some $n_h \in \mathbb{N}$ and $\tilde{C} > 0$ that does only depend on the diameter of $\mathscr{X}$ (not on h nor on μ) so that for every $n \geq n_h$, for every measure $\mu \in \mathscr{P}(\mathscr{X})$ so that $\inf_{x\in\mathscr{X}} \mathrm{d}_{\mu,h}(x) \geq l_h$,*

$$\mathbb{E}\left[\mathcal{W}_1\left(\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h_{\#\hat{\boldsymbol{\mu}}_n}}, \mathrm{d}_{\mu,h_{\#\hat{\boldsymbol{\mu}}_n}}\right)\right] \leq \mathbb{E}\left[\mathcal{W}_2\left(\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h_{\#\hat{\boldsymbol{\mu}}_n}}, \mathrm{d}_{\mu,h_{\#\hat{\boldsymbol{\mu}}_n}}\right)\right] \leq \frac{\tilde{C}}{h l_h \sqrt{n}} \tag{S.5.3}$$

PROOF. Let $h \in (0,1]$. According to (Chazal, Massart and Michel, 2016, Proposition 1), also valid in the more general metric space $(\mathscr{X}, \mathrm{d})$, we get that for every $x \in \mathscr{X}$:

$$\left|\mathrm{d}^2_{\hat{\boldsymbol{\mu}}_n,h}(x) - \mathrm{d}^2_{\mu,h}(x)\right| \leq \frac{1}{h}\mathcal{W}_1\left(\mathrm{d}^2(x,\cdot)_{\#\mu}, \mathrm{d}^2(x,\cdot)_{\#\hat{\boldsymbol{\mu}}_n}\right). \tag{S.5.4}$$

Moreover, according to (Bobkov and Ledoux, 2019, Theorem 3.2) and the discussion below, and since $(\mathscr{X}, \mathrm{d})$ is a compact set, we get both that

$$\sup_{\mu\in\mathscr{P}(\mathscr{X})} \sup_{x\in\mathscr{X}} \mathbb{E}_\mu\left[\mathcal{W}_1\left(\mathrm{d}^2(x,\cdot)_{\#\mu}, \mathrm{d}^2(x,\cdot)_{\#\hat{\boldsymbol{\mu}}_n}\right)\right] \leq \frac{C}{\sqrt{n}} \tag{S.5.5}$$

and

$$\sup_{\mu\in\mathscr{P}(\mathscr{X})} \sup_{x\in\mathscr{X}} \sqrt{\mathbb{E}_\mu\left[\mathcal{W}_1^2\left(\mathrm{d}^2(x,\cdot)_{\#\mu}, \mathrm{d}^2(x,\cdot)_{\#\hat{\boldsymbol{\mu}}_n}\right)\right]} \leq \frac{C}{\sqrt{n}} \tag{S.5.6}$$

for some positive constant C.

Besides, note that for $n > \frac{1}{h}$, for $\hat{\boldsymbol{\mu}}_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ and $\hat{\boldsymbol{\mu}}_{n-1} = \frac{1}{n-1}\sum_{i=2}^n \delta_{X_i}$, we get that

$$\mathrm{d}^2_{\hat{\boldsymbol{\mu}}_n,h}(X_1) = \mathrm{d}^2_{\hat{\boldsymbol{\mu}}_{n-1},\tilde{h}}(X_1), \tag{S.5.7}$$

where $\tilde{h} = \left(h - \frac{1}{n}\right)\frac{n}{n-1} = \frac{hn-1}{n-1}$, since X_1 is the nearest neighbour of X_1 in $\{X_1, \ldots, X_n\}$ with a distance equal to 0. Notice that X_1 is independent from $\hat{\boldsymbol{\mu}}_{n-1}$.

Moreover, note that for every $x \in \mathscr{X}$,

$$\left|\mathrm{d}^2_{\mu,h}(x) - \mathrm{d}^2_{\mu,\tilde{h}}(x)\right| \leq \left|\frac{1}{h}\int_{l=0}^{h} \delta^2_{\mu,l}(x) - \frac{1}{\tilde{h}}\int_{l=0}^{\tilde{h}} \delta^2_{\mu,l}(x)\right| \tag{S.5.8}$$

$$\leq \frac{2\mathscr{D}(\mathscr{X})^2|h-\tilde{h}|}{\tilde{h}} \tag{S.5.9}$$

$$\leq \frac{4\mathscr{D}(\mathscr{X})^2|h-\tilde{h}|}{h}, \tag{S.5.10}$$

according to (2.1), where the last inequality is correct for $n \geq n_h := \max\left(2, \frac{4}{h}\right)$ since

$$\tilde{h} = h - \frac{1-h}{n-1} \geq h - \frac{2}{n} \geq \frac{h}{2}. \tag{S.5.11}$$

As a consequence, if $\inf_{x\in\mathscr{X}} \mathrm{d}_{\mu,h}(x) \geq l_h > 0$ and $n \geq n_h$, then we get that:

$$\mathbb{E}\left[\mathcal{W}_2\left(\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h_{\#\hat{\boldsymbol{\mu}}_n}}, \mathrm{d}_{\mu,h_{\#\hat{\boldsymbol{\mu}}_n}}\right)\right] \leq \sqrt{\mathbb{E}\left[\mathcal{W}_2^2\left(\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h_{\#\hat{\boldsymbol{\mu}}_n}}, \mathrm{d}_{\mu,h_{\#\hat{\boldsymbol{\mu}}_n}}\right)\right]}$$

$$
\begin{aligned}
&\leq \sqrt{\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left|\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h}(X_i)-\mathrm{d}_{\mu,h}(X_i)\right|^2\right]}\\
&= \sqrt{\mathbb{E}\left[\left|\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h}(X_1)-\mathrm{d}_{\mu,h}(X_1)\right|^2\right]}\\
&\leq \sqrt{\mathbb{E}\left[\frac{\left|\mathrm{d}^2_{\hat{\boldsymbol{\mu}}_n,h}(X_1)-\mathrm{d}^2_{\mu,h}(X_1)\right|^2}{\left|\mathrm{d}_{\hat{\boldsymbol{\mu}}_n,h}(X_1)+\mathrm{d}_{\mu,h}(X_1)\right|^2}\right]}\\
&\leq \frac{1}{l_h}\sqrt{\mathbb{E}\left[\left|\mathrm{d}^2_{\hat{\boldsymbol{\mu}}_n,h}(X_1)-\mathrm{d}^2_{\mu,h}(X_1)\right|^2\right]}\\
&\leq \frac{1}{l_h}\sqrt{\mathbb{E}_{X_1\sim\mu}\mathbb{E}_{X_2,\ldots X_n\text{ iid}\sim\mu}\left[\left|\mathrm{d}^2_{\hat{\boldsymbol{\mu}}_{n-1},\tilde{h}}(X_1)-\mathrm{d}^2_{\mu,\tilde{h}}(X_1)\right|^2\right]}\\
&+\frac{1}{l_h}\sqrt{\mathbb{E}\left[\left|\mathrm{d}^2_{\mu,h}(X_1)-\mathrm{d}^2_{\mu,\tilde{h}}(X_1)\right|^2\right]}\\
&\leq \frac{C}{\tilde{h}l_h\sqrt{n-1}}+\frac{4\mathscr{D}(\mathscr{X})^2}{hl_h(n-1)}\\
&\leq \frac{\tilde{C}}{hl_h\sqrt{n}},
\end{aligned}
$$

for some constant $\tilde{C}>0$ that does not depend on μ nor on h, according to the Minkowski inequality, to (S.5.7), (S.5.8), (S.5.4) and (S.5.6), since X_1 is independent from $X_2,\ldots,X_n$. □

S.5.2.3. *Proof for Proposition 2.3 in Brécheteau (2025).* If $\mathscr{X}$ is discrete, this result is a consequence of Proposition S.2.3 below.

If $\mathscr{X}$ is not discrete, this result is a consequence of the unicity of uniform (resp. h_0-uniform) Borel probability measures Christensen (1970) (cf. Theorem S.1.1).

Let $H>0$, let $\mu\in\mathscr{P}(\mathscr{X})$ be such that $\forall h\in[0,H]$, $s_h(\mu)=s_h(\mu_0)$. Then, $\mu(\mathrm{B}(x,r))=\mu_0(\mathrm{B}(x,r))=\mu_0(\mathrm{B}(y,r))=\mu(\mathrm{B}(x,r))$ for every $x,y\in\mathscr{X}$ and $r\leq r(\min(h_0,H))$. Notice that $r(\min(h_0,H))>0$ since $(\mathscr{X},\mathrm{d})$ is not discrete. Consequently, μ is $\min(h_0,H)$-homogeneous, and by unicity, μ is thus equal to μ_0, that is also $\min(h_0,H)$-homogeneous.

S.5.2.4. *Proof for Proposition 2.4 in Brécheteau (2025).* For the sake of contradiction, assume that for every H, the number of accumulation points of $\{h\in[0,1],\ s_h(\mu)=s_h(\mu_0)\}$ is infinite in $(0,H]$, then there exists a sequence $(r_n)_n$ converging to 0 so that $\mu(\mathrm{B}(x,r_n))=\mu_0(\mathrm{B}(x,r_n))$ for every $n\in\mathbb{N}$ and every $x\in\mathscr{X}$. Following the proof of Theorem S.1.1, it implies that $\mu=\mu_0$.

S.5.2.5. *Proof for Proposition 2.5 in Brécheteau (2025).* The proof is similar to the proof of Proposition S.2.2 below, in Section S.6.2.2.

Let $l,h\in(0,1)$ and $x\in A_{l,h,\mu_1}$. Then, $\mathrm{d}_{\mu_l,h}(x)=\mathrm{d}_{\mu_0,\frac{h}{1-l}}(x)=\mathrm{d}_{\frac{h}{1-l}}$ for instance using (2.1) and the fact that $\delta_{\mu_l,m}(x)=\delta_{\mu_0,\frac{m}{1-l}}(x)$ for every $m<h$.

Then, since $A_{l,h,\mu_1} \subset \mathrm{Supp}(\mu_1)^c$,

$$\mathcal{W}_1(s_h(\mu_l), s_h(\mu_0)) \geq \mu_l(A_{l,h,\mu_1}) \left| \mathrm{d}_{\frac{h}{1-l}} - \mathrm{d}_h \right| = (1-l)\mu_0(A_{l,h,\mu_1}) \left| \mathrm{d}_{\frac{h}{1-l}} - \mathrm{d}_h \right|. \tag{S.5.12}$$

The equivalence follows from the fact that $f : h \mapsto \mathrm{d}_h$ is differentiable at h. So $\left| \mathrm{d}_{\frac{h}{1-l}} - \mathrm{d}_h \right| = \frac{l}{1-l}(hf'(h)) + O(l^2)$, so $\left| \mathrm{d}_{\frac{h}{1-l}} - \mathrm{d}_h \right| \sim_{l\to 0} C_h \mu_0(A_{l,h,\mu_1}) l$ for some constant non negative constant C_h that is equal to zero if and only if $\mathscr{X}$ is discrete with $h < \frac{1}{|\mathscr{X}|}$. Moreover, since the sets $(A_{l,h,\mu_1})_{l\geq 0}$ are non increasing for the inclusion, $\mu_0\left(A_{l,h,\mu_1}\right) \to_{l\to 0} \mu_0\left(\bigcup_{l>0} A_{l,h,\mu_1}\right) \in (0,1]$.

S.5.2.6. *Proof for Proposition 2.6 in Brécheteau (2025).* According to (Carlier, Delalande and Mérigot, 2024, Section 1.2.2),

$$\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{\mu}}_n), \bar{s}_h(\hat{\mathbb{\nu}}_n)) \leq \mathcal{W}_1(s_h(\hat{\mathbb{\mu}}_n), s_h(\hat{\mathbb{\nu}}_n)), \tag{S.5.13}$$

where $\mathcal{W}_1$ is computed with respect to the L_2-Wasserstein distance on $\mathscr{P}([0, \mathscr{D}(\mathscr{X})])$.

Then, using the optimal transport plan π between μ and ν for the L_2-Wasserstein distance, as in the proof of Theorem 1.1 in Brécheteau (2025), and using Proposition 2.1 in Brécheteau (2025), we get that:

$$\begin{aligned}
\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{\mu}}_n), \bar{s}_h(\hat{\mathbb{\nu}}_n)) &\leq \mathbb{E}_\pi\left[\mathcal{W}_1(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\hat{\boldsymbol{\nu}}_n))\right] \\
&\leq \mathbb{E}_\pi\left[\left(1 + \frac{1}{\sqrt{h}}\right)\mathcal{W}_2(\hat{\boldsymbol{\mu}}_n, \hat{\boldsymbol{\nu}}_n)\right] \\
&\leq \left(1 + \frac{1}{\sqrt{h}}\right)\mathcal{W}_2(\mu, \nu).
\end{aligned}$$

The second inequality is obtained with the same procedure, where $\mathbb{\pi}$ is the optimal transport map between $\mathbb{\mu}_n$ and $\mathbb{\nu}_n$ for the L_1-Wasserstein distance:

$$\begin{aligned}
\mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_n), \bar{s}_h(\mathbb{\nu}_n)) &\leq \mathbb{E}_{(\boldsymbol{\mu}_n, \boldsymbol{\nu}_n)\sim\mathbb{\pi}}\left[\left(1 + \frac{1}{\sqrt{h}}\right)\mathcal{W}_2(\boldsymbol{\mu}_n, \boldsymbol{\nu}_n)\right] \\
&= \left(1 + \frac{1}{\sqrt{h}}\right)\mathcal{W}_1(\mathbb{\mu}_n, \mathbb{\nu}_n).
\end{aligned}$$

S.5.2.7. *Proof for Proposition 2.7 in Brécheteau (2025).* Using triangular inequality and the definition of $\bar{s}_h(\hat{\mathbb{\mu}}_n)$, we get that:

$$\begin{aligned}
\mathcal{W}_1\left(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n)\right) &\leq \mathcal{W}_2\left(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n)\right) \\
&\leq \mathbb{E}_\mu\left[\mathcal{W}_2(s_h(\mu), s_h(\hat{\boldsymbol{\mu}}_n))\right] + \mathbb{E}_\mu\left[\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{\mu}}_n), s_h(\hat{\boldsymbol{\mu}}_n))\right] \\
&\leq 2\mathbb{E}_\mu\left[\mathcal{W}_2(s_h(\mu), s_h(\hat{\boldsymbol{\mu}}_n))\right].
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbb{E}[\mathcal{W}_1\left(s_h(\hat{\boldsymbol{\mu}}_n), \bar{s}_h(\hat{\mathbb{\mu}}_n)\right)] &\leq \mathbb{E}[\mathcal{W}_2\left(s_h(\hat{\boldsymbol{\mu}}_n), \bar{s}_h(\hat{\mathbb{\mu}}_n)\right)] \\
&\leq \mathbb{E}[\mathcal{W}_2\left(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu)\right)] + \mathcal{W}_2\left(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n)\right) \\
&\leq 3\mathbb{E}_\mu\left[\mathcal{W}_2(s_h(\mu), s_h(\hat{\boldsymbol{\mu}}_n))\right].
\end{aligned}$$

We conclude with Proposition 2.1 in Brécheteau (2025) and Theorem 1.1 in Brécheteau (2025).

The rates are obtained as a direct consequence of Proposition 2.2 in Brécheteau (2025).

S.5.2.8. *Proof for Proposition 2.8 in Brécheteau (2025).* Let $h \in (0,1)$, $l \in (0,1)$, $\mathbb{\mu}_{1,n} \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ and $\mathbb{\mu}_{l,n} = l\mathbb{\mu}_{1,n} + (1-l)\hat{\mathbb{\mu}}_{0,n}$. Then,

$$\begin{aligned}
\mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_{l,n}), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})) &= \sqrt{\int_{u=0}^{1} \left|Q_{\bar{s}_h(\mathbb{\mu}_{l,n})}(u) - Q_{\bar{s}_h(\hat{\mathbb{\mu}}_{0,n})}(u)\right|^2 \mathrm{d}u} \\
&= \sqrt{\int_{u=0}^{1} \left|\mathbb{E}_{\mathbb{s}\sim s_{h\#\mathbb{\mu}_{l,n}}}\left[Q_{\mathbb{s}}(u)\right] - \mathbb{E}_{\mathbb{s}\sim s_{h\#\hat{\mathbb{\mu}}_{0,n}}}\left[Q_{\mathbb{s}}(u)\right]\right|^2 \mathrm{d}u} \\
&= \sqrt{\int_{u=0}^{1} \left|\mathbb{E}_{\boldsymbol{\mu}\sim\mathbb{\mu}_{l,n}}\left[Q_{s_h(\boldsymbol{\mu})}(u)\right] - \mathbb{E}_{\boldsymbol{\mu}\sim\hat{\mathbb{\mu}}_{0,n}}\left[Q_{s_h(\boldsymbol{\mu})}(u)\right]\right|^2 \mathrm{d}u} \\
&= l\sqrt{\int_{u=0}^{1} \left|\mathbb{E}_{\boldsymbol{\mu}\sim\mathbb{\mu}_{1,n}}\left[Q_{s_h(\boldsymbol{\mu})}(u)\right] - \mathbb{E}_{\boldsymbol{\mu}\sim\hat{\mathbb{\mu}}_{0,n}}\left[Q_{s_h(\boldsymbol{\mu})}(u)\right]\right|^2 \mathrm{d}u} \\
&= l\mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_{1,n}), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})).
\end{aligned}$$

where $s_{h\#\mathbb{\mu}_n}$ is the distribution of $s_h(\boldsymbol{\mu})$ for a $\mathscr{P}_n(\mathscr{X})$-valued random measure $\boldsymbol{\mu}$ with distribution $\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X}))$.

S.5.3. *Proofs for Section 3 in Brécheteau (2025).*

S.5.3.1. *Proof for Theorem 3.1 in Brécheteau (2025).*

S.5.3.1.1. Proof of the consistency given by (3.17) for the tests of homogeneity.. Let $h \in (0,1]$. Let $\mathrm{c} > 0$ and $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0. Recall that:

$$\mathrm{H}_1(\hat{\mathbb{\mu}}_{0,n}, h, \mathrm{c}, \epsilon_n) = \{\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\ \mathcal{W}_2(\bar{s}_h(\mathbb{\mu}_n), \bar{s}_h(\hat{\mathbb{\mu}}_{0,n})) > \mathrm{c},\ \mathcal{V}_{2,h}(\mathbb{\mu}_n) \leq \epsilon_n\}.$$

Since $\mathscr{X}$ is compact, the distances $(\mathcal{W}_p)_{p\geq 1}$ are comparable. Let $w_{\mathscr{X}} > 0$ be such that:

$$\forall \mu, \nu \in \mathscr{P}([0, \mathscr{D}(\mathscr{X})]),\ \mathcal{W}_1(\mu,\nu) \leq \mathcal{W}_2(\mu,\nu) \leq w_{\mathscr{X}}\sqrt{\mathcal{W}_1(\mu,\nu)}. \tag{S.5.14}$$

According to (3.21), the quantile of $\mathbf{T}_{n,h}^{\mathrm{hom}}$ under H_0 satisfies $\mathrm{q}_{1-\alpha,\mathrm{n},\mathrm{h}}^{\mathrm{hom}} \to 0$ when $n \to \infty$, and the barycenter signature of $\hat{\mathbb{\mu}}_{0,n}$ converges to the signature of μ_0, according to Proposition 2.7 in Brécheteau (2025). Let $n_{\mathrm{c}} \in \mathbb{N}$ be such that $\mathrm{q}_{1-\alpha,\mathrm{n},\mathrm{h}}^{\mathrm{hom}} \leq \left(\frac{\mathrm{c}}{2\mathrm{w}_{\mathscr{X}}}\right)^2$ and so that $\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{\mu}}_{0,n}), s_h(\mu_0)) \leq \frac{\mathrm{c}}{4}$ for every $n \geq n_{\mathrm{c}}$. Then, for every $n \geq n_{\mathrm{c}}$ and every $\mathbb{\mu}_n \in \mathrm{H}_1(\hat{\mathbb{\mu}}_{0,n}, h, \mathrm{c}, \epsilon_n)$,

$$\begin{aligned}
&\mathbb{P}_{\mathbb{\mu}_n}\left(\mathbf{T}_{n,h}^{\mathrm{hom}} \geq \mathrm{q}_{1-\alpha,\mathrm{n},\mathrm{h}}^{\mathrm{hom}}\right) \\
&\geq \mathbb{P}_{\mathbb{\mu}_n}\left(\mathbf{T}_{n,h}^{\mathrm{hom}} \geq \left(\frac{\mathrm{c}}{2w_{\mathscr{X}}}\right)^2\right) \\
&\geq \mathbb{P}_{\boldsymbol{\mu}_n\sim\mathbb{\mu}_n}\left(\mathcal{W}_2(s_h(\mu_0), s_h(\boldsymbol{\mu}_n)) \geq \frac{\mathrm{c}}{2}\right), \\
&\quad \text{according to (S.5.14) since all signatures are supported on } [0, \mathscr{D}(\mathscr{X})] \\
&\geq \mathbb{P}_{\boldsymbol{\mu}_n\sim\mathbb{\mu}_n}\left(\mathcal{W}_2\left(s_h(\mu_0), \bar{s}_h(\mathbb{\mu}_n)\right) - \frac{\mathrm{c}}{2} \geq \mathcal{W}_2\left(\bar{s}_h(\mathbb{\mu}_n), s_h(\boldsymbol{\mu}_n)\right)\right)
\end{aligned}$$

$$\geq \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} \left(\mathcal{W}_2 \left(\bar{s}_h(\hat{\mathbb{µ}}_{0,n}), \bar{s}_h(\mathbb{µ}_n)\right) - \frac{\mathrm{c}}{2} \geq \mathcal{W}_2 \left(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n)\right) + \mathcal{W}_2 \left(\bar{s}_h(\hat{\mathbb{µ}}_{0,n}), s_h(\mu_0)\right) \right)$$
$$\geq 1 - \sup_{\mathbb{µ}_n \in \mathrm{H}_1(\hat{\mathbb{µ}}_{0,n}, h, \mathrm{c}, \epsilon_n)} \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} (\mathcal{W}_2 \left(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n)\right) + \mathcal{W}_2 \left(\bar{s}_h(\hat{\mathbb{µ}}_{0,n}), s_h(\mu_0)\right)$$
$$> \mathcal{W}_2 \left(\bar{s}_h(\hat{\mathbb{µ}}_{0,n}), \bar{s}_h(\mathbb{µ}_n)\right) - \frac{\mathrm{c}}{2})$$
$$\geq 1 - \sup_{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\, \mathbb{E}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n}[\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n))] \leq \epsilon_n} \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} \left(\mathcal{W}_2 \left(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n)\right) > \frac{\mathrm{c}}{4} \right).$$
$$\geq 1 - \frac{4\epsilon_n}{c}, \text{ according to the Markov inequality.}$$

For a sequence $(\mathbb{µ}_n)_{n\in\mathbb{N}}$ of measures in $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ such that

$$\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), \bar{s}_h(\hat{\mathbb{µ}}_{0,n})) \geq c > 0$$

for every $n \in \mathbb{N}$, and such that $\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n))$ converges in probability to 0, then,

$$\mathbb{P}_{\mathbb{µ}_n} \left(\mathbf{T}^{\mathrm{hom}}_{n,h} \geq \mathrm{q}^{\mathrm{hom}}_{1-\alpha,\mathrm{n},\mathrm{h}} \right) \geq 1 - \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} \left(\mathcal{W}_2 \left(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n)\right) > \frac{\mathrm{c}}{4} \right), \tag{S.5.15}$$

for $n \geq n_{\mathrm{c}}$, so, the power converges to 1 when n goes to $+\infty$.

S.5.3.1.2. Proof of the consistency given by (3.17) *for the tests of iidness..* We follow the proof of the previous paragraph for $\phi^{\mathrm{hom}}_{n,h}$. Let $h \in (0,1]$. According to (3.21), the quantile of $\mathbf{T}^{\mathrm{iid}}_{n,h}$ under H_0 satisfies $\mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n},\mathrm{h}} \to 0$ when $n \to \infty$. Let $n_{\mathrm{c}} \in \mathbb{N}$ be such that $\mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n},\mathrm{h}} \leq \frac{\mathrm{c}}{2}$ for every $n \geq n_{\mathrm{c}}$. Then, for every $n \geq n_{\mathrm{c}}$ and every $\mathbb{µ}_n \in \mathrm{H}_1(\hat{\mathbb{µ}}_{0,n}, h, \mathrm{c}, \epsilon_n)$,

$$\mathbb{P}_{\mathbb{µ}_n} \left(\mathbf{T}^{\mathrm{iid}}_{n,h} \geq \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n},\mathrm{h}} \right) \geq \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} \left(\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{µ}}_{0,n}), s_h(\boldsymbol{\mu}_n)) \geq \frac{\mathrm{c}}{2} \right)$$
$$\geq \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} \left(\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{µ}}_{0,n}), \bar{s}_h(\mathbb{µ}_n)) - \frac{\mathrm{c}}{2} \geq \mathcal{W}_2 \left(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n)\right) \right)$$
$$\geq 1 - \sup_{\mathbb{µ}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\, \mathbb{E}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n}[\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n))] \leq \epsilon_n} \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} \left(\mathcal{W}_2 \left(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n)\right) > \frac{\mathrm{c}}{2} \right)$$
$$\geq 1 - \frac{2\epsilon_n}{c}.$$

For a sequence $(\mathbb{µ}_n)_{n\in\mathbb{N}}$ of measures in $\mathscr{P}(\mathscr{P}_n(\mathscr{X}))$ such that $\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), \bar{s}_h(\hat{\mathbb{µ}}_{0,n})) \geq c > 0$ for every $n \in \mathbb{N}$, and such that $\mathcal{W}_2(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n))$ converges in probability to 0, then,

$$\mathbb{P}_{\mathbb{µ}_n} \left(\mathbf{T}^{\mathrm{iid}}_{n,h} \geq \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n},\mathrm{h}} \right) \geq 1 - \mathbb{P}_{\boldsymbol{\mu}_n \sim \mathbb{µ}_n} \left(\mathcal{W}_2 \left(\bar{s}_h(\mathbb{µ}_n), s_h(\boldsymbol{\mu}_n)\right) > \frac{\mathrm{c}}{2} \right), \tag{S.5.16}$$

for $n \geq n_{\mathrm{c}}$, so, the power converges to 1 when n goes to $+\infty$.

S.5.3.1.3. Proof of the consistency given by (3.18)*:.* Let $\mathrm{c} > 0$. Let $\mu \in \mathscr{P}(\mathscr{X})$ such that

$$\mathcal{W}_1(s_h(\mu), s_h(\mu_0)) > \mathrm{c}. \tag{S.5.17}$$

Let $\epsilon \in (0, \mathrm{c})$ and let $n_\epsilon \in \mathbb{N}$ be such that

$$\sup_{\mu \in \mathscr{P}(\mathscr{X}), n \geq n_\epsilon} \mathcal{W}_2(\bar{s}_h(\hat{\mathbb{µ}}_n), s_h(\mu)) \leq \frac{\epsilon}{2}, \tag{S.5.18}$$

given by Proposition 2.7 in Brécheteau (2025). Then, using the triangular inequality for $\mathcal{W}_2$, we get that

$$\mathcal{W}_2(\bar{s}_h(\hat{\mathbb{µ}}_n), \bar{s}_h(\hat{\mathbb{µ}}_{0,n})) > c - \epsilon. \tag{S.5.19}$$

Moreover, for $n \in \mathbb{N}$, let

$$\epsilon_n = \sup_{\mu \in \mathscr{P}(\mathscr{X})} \mathbb{E}\left[\mathcal{W}_2(s_h(\mu), s_h(\hat{\mu}_n)\right] + \sup_{\mu \in \mathscr{P}(\mathscr{X})} \mathbb{E}\left[\mathcal{W}_2(s_h(\mu), \bar{s}_h(\hat{\mathbb{\mu}}_n))\right]. \tag{S.5.20}$$

According to Proposition 2.1, Theorem 1.1 and Proposition 2.7 in Brécheteau (2025), the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converges to 0.

Then, (3.18) follows from (3.17), with the constant $\mathrm{c} - \epsilon > 0$ and the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ above defined.

S.5.3.1.4. Proof of the consistency given by (3.19)*:.* For every $\mathrm{c} > 0$, the set of measures $\{\mu \in \mathscr{P}(\mathscr{X}), \mathcal{W}_2(\mu, \mu_0) \geq \mathrm{c}\}$ is a compact set, and the function $\mathrm{c} \mapsto \mathcal{W}_1(s_h(\mu), s_h(\mu_0))$ is continuous according to Proposition 2.1 in Brécheteau (2025). Consequently, it is minimised, with a minimum that cannot be 0 since $h \in \mathscr{H}(\mathscr{X})$. According to Proposition S.2.3, this is the case for homogeneous discrete compact Polish spaces, for parameters $h \in \mathscr{H}(\mathscr{X})$, with $\mathscr{H}(\mathscr{X})$ that is not empty.

S.5.3.2. *Proof for Theorem 3.2 in Brécheteau (2025).*

S.5.3.2.1. Case $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_1, h, \epsilon n^{-\frac{1}{2}}, \mathrm{c}n^{-\frac{1}{2}}\right)$*, for* $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{hom}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{hom}}_{1-\alpha,\mathrm{n,h}})$*:.* We follow the proof of Theorem 3.1 in Brécheteau (2025). Let $h \in (0,1]$ and $\epsilon > 0$. For $n \in \mathbb{N}$, let $\bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n})$ be a median of $s_h(\hat{\boldsymbol{\mu}}_{0,n})$ for $\hat{\boldsymbol{\mu}}_{0,n} \sim \hat{\mathbb{\mu}}_{0,n}$, with the same proof as for (2.27), we get that $\mathcal{W}_1(s_h(\mu_0), \bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n})) \leq \frac{C}{\mathrm{hd}_h\sqrt{n}}$, for some $C > 0$, for $n \geq n_h$. Moreover, $\mathrm{q}^{\mathrm{hom}}_{1-\alpha,\mathrm{n,h}} \leq \frac{\mathrm{c}_{\alpha,\mathrm{h}}}{\sqrt{\mathrm{n}}} = \frac{\mathrm{C}}{\mathrm{hd_h}\alpha\sqrt{\mathrm{n}}}$ according to (3.21). So,

$$\begin{aligned}
&\mathbb{P}_{\mathbb{\mu}_n}\left(\mathbf{T}^{\mathrm{hom}}_{n,h} \geq \mathrm{q}^{\mathrm{hom}}_{1-\alpha,\mathrm{n,h}}\right)\\
&\geq \mathbb{P}_{\mathbb{\mu}_n}\left(\mathcal{W}_1\left(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n}), \bar{s}_{h,1}(\mathbb{\mu}_n)\right) \geq \frac{\mathrm{c}_{\alpha,h}}{\sqrt{n}} + \mathcal{W}_1\left(\bar{s}_{h,1}(\mathbb{\mu}_n), s_h(\boldsymbol{\mu}_n)\right) + \mathcal{W}_1\left(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n}), s_h(\mu_0)\right)\right)\\
&\geq \mathbb{P}_{\mathbb{\mu}_n}\left(\mathcal{W}_1\left(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n}), \bar{s}_{h,1}(\mathbb{\mu}_n)\right) \geq 2\frac{\mathrm{c}_{\alpha,h}}{\sqrt{n}} + \frac{\epsilon}{\beta\sqrt{n}}\right) - \mathbb{P}\left(\mathcal{W}_1\left(\bar{s}_{h,1}(\mathbb{\mu}_n), s_h(\boldsymbol{\mu}_n)\right) > \frac{\epsilon}{\beta\sqrt{n}}\right)\\
&\geq 1 - \beta,
\end{aligned}$$

provided that $\mathbb{E}\left[\mathcal{W}_1\left(\bar{s}_{h,1}(\mathbb{\mu}_n), s_h(\boldsymbol{\mu}_n)\right)\right] \leq \frac{\epsilon}{\sqrt{n}}$, according to the Markov inequality. We shall take $C = 2\mathrm{c}_{\alpha,h} + \frac{\epsilon}{\beta}$. align*

S.5.3.2.2. Case $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_2, h, \epsilon n^{-\frac{1}{r}}, \mathrm{c}n^{-\frac{1}{r}}\right)$*, for* $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{iid}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n,h}})$*:.* Let $h \in (0,1]$ and $\epsilon > 0$. For $n \geq n_h$, since $\mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n,h}} \leq \frac{\mathrm{c}_{\alpha,\mathrm{h}}}{\sqrt{\mathrm{n}}} = \frac{\mathrm{C}}{\mathrm{hd_h}\alpha\sqrt{\mathrm{n}}}$ according to (3.21), for some $C > 0$, we have that:

$$\begin{aligned}
&\mathbb{P}_{\mathbb{\mu}_n}\left(\mathbf{T}^{\mathrm{iid}}_{n,h} \geq \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n,h}}\right)\\
&\geq \mathbb{P}_{\mathbb{\mu}_n}\left(\mathcal{W}_2\left(\bar{s}_h(\hat{\mathbb{\mu}}_{0,n}), \bar{s}_h(\mathbb{\mu}_n)\right) \geq \frac{\mathrm{c}_{\alpha,h}}{\sqrt{n}} + \mathcal{W}_2\left(\bar{s}_h(\mathbb{\mu}_n), s_h(\boldsymbol{\mu}_n)\right)\right)\\
&\geq \mathbb{P}_{\mathbb{\mu}_n}\left(\mathcal{W}_2\left(\bar{s}_h(\hat{\mathbb{\mu}}_{0,n}), \bar{s}_h(\mathbb{\mu}_n)\right) \geq \frac{\mathrm{c}_{\alpha,h}}{\sqrt{n}} + \frac{\epsilon}{\beta\sqrt{n}}\right) - \mathbb{P}\left(\mathcal{W}_2\left(\bar{s}_h(\mathbb{\mu}_n), s_h(\boldsymbol{\mu}_n)\right) > \frac{\epsilon}{\beta\sqrt{n}}\right)\\
&\geq 1 - \beta,
\end{aligned}$$

provided that $\mathbb{E}\left[\mathcal{W}_2\left(\bar{s}_h(\mathbb{\mu}_n), s_h(\boldsymbol{\mu}_n)\right)\right] \leq \frac{\mathrm{c}_{\alpha,h}\beta}{\sqrt{n}}$, according to the Markov inequality. We shall take $C = \mathrm{c}_{\alpha,h} + \frac{\epsilon}{\beta}$. The proof is still valid when we replace $\sqrt{n}$ by $n^{\frac{1}{r}}$ for some $r \geq 2$.

S.5.3.2.3. Case $\mathrm{H}_1\left(\mu_0, \mathcal{W}_1, l_h, \mathrm{c}n^{-\frac{1}{2}}\right)$*, for* $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{hom}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{hom}}_{1-\alpha,\mathrm{n,h}})$*:.* This case is a direct consequence of the case $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_1, h, \epsilon n^{-\frac{1}{2}}, \mathrm{c}n^{-\frac{1}{2}}\right)$, for $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{hom}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{hom}}_{1-\alpha,\mathrm{n,h}})$, with $\epsilon = \frac{C}{hl_h}$.

Indeed, according to the definition of the median and (2.14),

$$\begin{aligned}\mathbb{E}\left[\mathcal{W}_1\left(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_n), s_h(\hat{\boldsymbol{\mu}}_n)\right)\right] &\leq \mathbb{E}\left[\mathcal{W}_1\left(s_h(\mu), s_h(\hat{\boldsymbol{\mu}}_n)\right)\right] \\ &\leq \frac{C}{hl_h\sqrt{n}}.\end{aligned}$$

Moreover, since according to (2.14), (2.18) and the definition of the median,

$$\begin{aligned}\mathcal{W}_1\left(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n}), \bar{s}_{h,1}(\hat{\mathbb{\mu}}_n)\right) &\geq \mathcal{W}_1(s_h(\mu_0), s_h(\mu)) - \mathcal{W}_1(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n}), s_h(\mu_0)) - \mathcal{W}_1(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_n), s_h(\mu)) \\ &\geq \mathcal{W}_1(s_h(\mu_0), s_h(\mu)) \\ &\quad -2\mathbb{E}\left[\mathcal{W}_1(s_h(\hat{\boldsymbol{\mu}}_{0,n}), s_h(\mu_0))\right] - 2\mathbb{E}\left[\mathcal{W}_1(s_h(\hat{\boldsymbol{\mu}}_n), s_h(\mu))\right] \\ &\geq \mathcal{W}_1(s_h(\mu_0), s_h(\mu)) - 4\frac{C}{h\min(l_h, d_h)\sqrt{n}},\end{aligned}$$

we get that:

$$\begin{aligned}&\left\{\hat{\mathbb{\mu}}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\, \mu \in \mathscr{P}(\mathscr{X}),\, \mathcal{W}_1(s_h(\mu), s_h(\mu_0)) \geq \frac{\mathrm{c}}{\sqrt{n}}\right\} \\ &\subset \left\{\mathbb{\mu}_n \in \mathscr{P}(\mathscr{P}_n(\mathscr{X})),\, \mathcal{W}_1\left(\bar{s}_{h,1}(\hat{\mathbb{\mu}}_{0,n}), \bar{s}_{h,1}(\mathbb{\mu}_n)\right) \geq \frac{\mathrm{c}}{\sqrt{n}} - 4\frac{C}{h\min(l_h, d_h)\sqrt{n}}\right\}\end{aligned}$$

with $\frac{\mathrm{c}}{\sqrt{n}} - 4\frac{C}{h\min(l_h,d_h)\sqrt{n}} \geq \frac{\tilde{C}}{\sqrt{n}}$ with $\tilde{C} = 2\mathrm{c}_{\alpha,h} + \frac{\epsilon}{\beta}$ (the constant C for the case $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_1, h, \epsilon n^{-\frac{1}{2}}, \mathrm{c}n^{-\frac{1}{2}}\right)$), for c large enough.

S.5.3.2.4. Case $\mathrm{H}_1\left(\mu_0, \mathcal{W}_2, l_h, \mathrm{c}n^{-\frac{1}{4}}\right)$*, for* $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{iid}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n,h}})$*:.* As above, this case is a direct consequence of the case $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_2, h, \epsilon n^{-\frac{1}{r}}, \mathrm{c}n^{-\frac{1}{r}}\right)$, for $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{iid}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n,h}})$, together with (2.15) and (2.18).

S.5.3.2.5. Case $\mathrm{H}_1\left(\mu_0, \mathcal{W}_2, l_h, \mathrm{c}_J, \mathrm{c}n^{-\frac{1}{2}}\right)$*, for* $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{iid}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n,h}})$*:.* This case is a direct consequence of the case $\mathrm{H}_1\left(\hat{\mathbb{\mu}}_{0,n}, \mathcal{W}_2, h, \epsilon n^{-\frac{1}{r}}, \mathrm{c}n^{-\frac{1}{r}}\right)$, for $(\mathbf{T}_{n,h}, \mathrm{q}) = (\mathbf{T}^{\mathrm{iid}}_{\mathrm{n,h}}, \mathrm{q}^{\mathrm{iid}}_{1-\alpha,\mathrm{n,h}})$, together with (2.17) and (2.18).

S.5.3.3. *Proof for Theorem 3.3 in Brécheteau (2025).* We use the following well known Lemma S.5.2.

LEMMA S.5.2. *If $\mu_0^{\otimes n}$ is the uniform measure on $\mathscr{X}^n$ and $\mu^{(n)}$ is a Borel probability measure on $\mathscr{X}^n$. Let $\mu^{(n)} = \mu^{\mathrm{abs}} + \mu^{\mathrm{sing}}$ be its Lebesgue decomposition given by the Radon-Nikodym-Lebesgue theorem, with μ^{abs} absolutely continuous with respect to $\mu_0^{\otimes n}$, with Radon-Nikodym density f, and μ^{sing} singular with respect to $\mu_0^{\otimes n}$.*

Then, for every sequence of random variables $(\phi_n)_{n\in\mathbb{N}}$, with values in $\{0,1\}$, where $\phi_n = \phi_n(X_1, \ldots, X_n)$ is a function of a random vector $(X_1, \ldots, X_n) \in \mathscr{X}^n$:

$$\mathbb{P}_{(X_1,\ldots,X_n)\sim\mu_0^{\otimes n}}(\phi_n = 1) + \mathbb{P}_{(X_1,\ldots,X_n)\sim\mu^{(n)}}(\phi_n = 0)$$

$$\geq \int_{\mathscr{X}^n} \min(1, f(x_1,\ldots,x_n))\mathrm{d}\mu_0^{\otimes n}(x_1,\ldots,x_n).$$

PROOF. First, notice that

$$\int_{\mathscr{X}^n} \mathbb{1}_{\phi_n=1}(f-1)\mathrm{d}\mu_0^{\otimes n} + \int_{\mathscr{X}^n} \mathbb{1}_{\phi_n=1}\mathrm{d}\mu^{\mathrm{sing}} \leq \int_{\mathscr{X}^n} \mathbb{1}_{f\geq 1}(f-1)\mathrm{d}\mu_0^{\otimes n} + \mu^{\mathrm{sing}}\left(\mathscr{X}^n\right).$$

So,

$$\begin{aligned}\mathbb{P}_{\mu_0^{\otimes n}}(\boldsymbol{\phi}_n=1)+\mathbb{P}_{\mu^{(n)}}(\boldsymbol{\phi}_n=0) &\geq 1+\textstyle\int_{\mathscr{X}^n} \mathbb{1}_{f\geq 1}(1-f)\mathrm{d}\mu_0^{\otimes n} - \mu^{\mathrm{sing}}\left(\mathscr{X}^n\right)\\ &= \textstyle\int_{\mathscr{X}^n}\left(\mathbb{1}_{f\geq 1}(1-f)+f\right)\mathrm{d}\mu_0^{\otimes n}\\ &= \textstyle\int_{\mathscr{X}^n}\min(1,f)\mathrm{d}\mu_0^{\otimes n}\end{aligned}$$

□

This is a consequence of Proposition 2.5 in Brécheteau (2025) with $\mu_1=\delta_{x_0}$ together with Lemma S.5.2.

Let $\gamma\in(0,1)$ and $l_{\gamma,n}=\frac{-\log(1-\gamma)}{n}$. Let $\mu_{\gamma,n}=l_{\gamma,n}\mu_1+(1-l_{\gamma,n})\mu_0$. Then, for f the Radon-Nikodym density of $\mu_{\gamma,n}^{\otimes n,\mathrm{abs}}$, the sub-measure of $\mu_{\gamma,n}^{\otimes n}$ absolutely continuous with respect to $\mu_0^{\otimes n}$ given by the Radon-Nikodym-Lebesgue theorem, we have that:

$$\begin{aligned}\liminf_{n\to+\infty}\mathbb{P}_{\mu_0}(\boldsymbol{\phi}_n=1)+\mathbb{P}_{\mu_{\gamma,n}}(\boldsymbol{\phi}_n=0) &\geq \liminf_{n\to+\infty}\int_{\mathscr{X}^n}\min\left(1,f(x_1,\ldots,x_n)\right)\mathrm{d}\mu_0^{\otimes n}(x_1,\ldots,x_n)\\ &= \liminf_{n\to+\infty}\int_{\mathscr{X}^n}\min\left(1,(1-l_{\gamma,n})^n\right)\mathrm{d}\mu_0^{\otimes n}(x_1,\ldots,x_n)\\ &= \liminf_{n\to+\infty}\exp(n\log(1-l_{\gamma,n}))\\ &= 1-\gamma.\end{aligned}$$

Indeed, note that $f(x_1,\ldots,x_n)=(1-l_{\gamma,n})^n$ for every $(x_1,\ldots,x_n)\in(\mathscr{X}\backslash\{x_0\})^n$.

Moreover, according to Proposition 2.5,

$$\begin{aligned}\mathcal{W}_2(s_h(\mu_{\gamma,n}),s_h(\mu_0)) &\geq W_1(s_h(\mu_{\gamma,n}),s_h(\mu_0))\\ &\geq (1-l_{\gamma,n})\mu_0(A_{l_{\gamma,n},h,\delta_{x_0}})\left|\mathrm{d}_{\frac{h}{1-l_{\gamma,n}}}-\mathrm{d}_h\right|\\ &\sim_{n\to+\infty} C_{h,\delta_{x_0}}\frac{-\log(1-\gamma)}{n}.\end{aligned}$$

The Theorem follows from the choice $C=-\frac{1}{2}C_{h,\delta_{x_0}}\log(1-\gamma)>0$.

S.5.3.4. *Proofs for Proposition 4.1 in Brécheteau (2025).* The function $f:\mu_n\in(\mathscr{P}(\mathscr{X}),\mathcal{W}_2)\mapsto s_h(\mu_n)\in(\mathscr{P}([0,\mathscr{D}(\mathscr{X})]),\mathcal{W}_1)$, is continuous, according to Proposition 2.1 in Brécheteau (2025). Moreover, $\mathscr{P}_n(\mathscr{X})$ is a closed subset of the compact set $\mathscr{P}(\mathscr{X})$, according to Lemma S.5.3, and is therefore compact. So, the function f admits a minimum, and $\mathscr{P}_{n,h}^{\mathrm{opt}}(\mathscr{X})$ is not empty.

Moreover, if $\mathbb{μ}_n$ is supported on $\mathscr{P}_{n,h}^{\mathrm{opt}}(\mathscr{X})$ and $\boldsymbol{\mu}_n\sim\mathbb{μ}_n$, we get that for every $\mu_n\in\mathscr{P}_n(\mathscr{X})$,

$$\mathbf{T}_{n,h}^{\mathrm{hom}}=\mathcal{W}_1\left(s_h(\boldsymbol{\mu}_n),s_h(\mu_0)\right)\leq\mathcal{W}_1\left(s_h(\mu_n),s_h(\mu_0)\right). \tag{S.5.21}$$

So, $\mathbf{T}_{n,h}^{\mathrm{hom}}\leq \mathrm{q}_{1-\alpha,\mathrm{n},\mathrm{h}}^{\mathrm{hom}}$.

LEMMA S.5.3. *Let $n \in \mathbb{N}$, the subset $\mathscr{P}_n(\mathscr{X})$ of $\mathscr{P}(\mathscr{X})$ equipped with the Wasserstein distance $\mathcal{W}_2$ is closed.*

PROOF. Let $\left(P_m = \frac{1}{n}\sum_{i=1}^n \delta_{x_i^{(m)}}\right)_{m\in\mathbb{N}}$ be a sequence in $\mathscr{P}_n(\mathscr{X})$ converging weakly to some probability measure $P \in \mathscr{P}(\mathscr{X})$. Since $(\mathscr{X}, \mathrm{d})$ is compact, let $(x_i^*)_{1\le i\le n}$ be a limit of a subsequence of $\left(\left(x_i^{(m)}\right)_{1\le i\le n}\right)_{m\in\mathbb{N}}$, and $P^* = \frac{1}{n}\sum_{i=1}^n \delta_{x_i^*}$. Since $\mathcal{W}_2^2(P_m, P^*) \le \frac{1}{n}\sum_{i=1}^n \left|x_i^* - x_i^{(m)}\right|^2$, it follows that $P^* = P$, so that $P \in \mathscr{P}_n(\mathscr{X})$. $\square$

S.6. Proofs of the results in this supplement.

S.6.1. *Proofs for Section S.1 of the supplement.*

S.6.1.1. *Proof for Theorem S.1.1.* Let ϕ be a continuous fonction on $(\mathscr{X}, \mathrm{d})$. We shall prove that $\int_{\mathscr{X}} \phi(x)\mathrm{d}\mu(x) = \int_{\mathscr{X}} \phi(x)\mathrm{d}\nu(x)$. The conclusion would follow from (Dudley, 2002, Lemma 9.3.2).

Let μ (resp. ν) be an $h_0(\mu)$-uniform (resp. $h_0(\nu)$-uniform) measure.

For $0 < r < r^* := \min(\delta_{\mu,h_0(\mu)}, \delta_{\nu,h_0(\nu)})$ ($r^* > 0$ since $\min(h_0(\mu), h_0(\nu)) > l_0$), let $c_r = \mu(\mathrm{B}(x,r))$ and $c'_r = \nu(\mathrm{B}(x,r))$ for some (and thus for all) $x \in \mathscr{X}$, and let $K_r\phi(x) = \int_{\mathrm{B}(x,r)} \phi(y)\frac{\mathrm{d}\mu(y)}{c_r}$.

Firstly notice that according to the Fubini theorem, and since μ and ν satisfy (1.1) or (1.2),

$$\begin{aligned}(\text{S.6.1})\qquad &\int_{\mathscr{X}} K_r\phi(x)\mathrm{d}\mu(x) - \frac{c_r}{c'_r}\int_{\mathscr{X}} K_r\phi(x)\mathrm{d}\nu(x)\\ &= \int_{\mathscr{X}} \phi(y)\left(\int_{\mathscr{X}} \mathbb{1}_{\mathrm{d}(x,y)<r}\frac{\mathrm{d}\mu(x)}{c_r} - \frac{c_r}{c'_r}\int_{\mathscr{X}} \mathbb{1}_{\mathrm{d}(x,y)<r}\frac{\mathrm{d}\nu(x)}{c_r}\right)\mathrm{d}\mu(y) = 0.\end{aligned}$$

Secondly notice that,

$$\begin{aligned}(\text{S.6.2})\qquad &\left|\int_{\mathscr{X}} K_r\phi(x)\mathrm{d}\nu(x) - \int_{\mathscr{X}} \phi(x)\mathrm{d}\nu(x)\right|\\ &\le \int_{\mathscr{X}}\left(\int_{\mathscr{X}} \mathbb{1}_{\mathrm{B}(x,r)}(z)\,|\phi(z) - \phi(x)|\,\frac{\mathrm{d}\mu(z)}{c_r}\right)\mathrm{d}\nu(x) \le \omega_r(\phi) \to 0,\ r \to 0.\end{aligned}$$

where $\omega_r(\phi) = \sup_{x,y\in\mathscr{X},\mathrm{d}(x,y)<r} |\phi(x) - \phi(y)|$ converges to 0 when r converges to 0 since ϕ is continuous on the compact $\mathscr{X}$, and thus uniformly continuous.

According to (S.6.1) and (S.6.2), also true for μ, it follows that $\int_{\mathscr{X}} \phi(x)\mathrm{d}\mu(x) = \lambda\int_{\mathscr{X}} \phi(x)\mathrm{d}\nu(x)$, with $\lambda = \lim_{r\to 0}\frac{c_r}{c'_r}$ that does not depend on ϕ. Taking $\phi = 1$, we get that $\lambda = 1$ since μ and ν are probability measures.

The second part follows from the same proof taking $\mu = \mu_0$, $c'_r = c_r$ and considering radii r in the sequence $(r_n)_{n\in\mathbb{N}}$.

S.6.1.2. *Proof of Theorem S.1.2..* Let μ and ν be two Borel probability measures satisfying (S.1.3). Let μ_0 be a quasi-uniform measure. Let ϕ be a continuous fonction on $(\mathscr{X}, \mathrm{d})$. We shall prove that $\int_{\mathscr{X}} \phi(x)\mathrm{d}\mu(x) = \int_{\mathscr{X}} \phi(x)\mathrm{d}\nu(x)$. The conclusion would follow from (Dudley, 2002, Lemma 9.3.2) that states that two Borel probability measures μ and ν so that $\int \phi(x)\mathrm{d}\mu(x) = \int \phi(x)\mathrm{d}\nu(x)$ for every bounded continuous function ϕ coincide.

For $0<\epsilon<\epsilon_0$, and $x\in\mathscr{X}$, let $c_{\epsilon,x}=\mu_0(\mathrm{B}(x,\epsilon))$ and let $K_\epsilon\phi(x)=\int_{\mathrm{B}(x,\epsilon)}\phi(y)\frac{\mathrm{d}\mu_0(y)}{c_{\epsilon,x}}$. Firstly notice that

$$\int_{\mathscr{X}} K_\epsilon\phi(x)\mathrm{d}\mu(x)-\int_{\mathscr{X}} K_\epsilon\phi(x)\mathrm{d}\nu(x)\to 0,\ \epsilon\to 0, \tag{S.6.3}$$

Indeed,

$$\begin{aligned}
&\left|\int_{\mathscr{X}} K_\epsilon\phi(x)\mathrm{d}\mu(x)-\int_{\mathscr{X}} K_\epsilon\phi(x)\mathrm{d}\nu(x)\right| \\
&\leq \int_{\mathscr{X}}|\phi(y)|\left|\int_{\mathscr{X}}\mathbb{1}_{\mathrm{d}(x,y)<\epsilon}\frac{\mathrm{d}\mu(x)}{c_{x,\epsilon}}-\int_{\mathscr{X}}\mathbb{1}_{\mathrm{d}(x,y)<\epsilon}\frac{\mathrm{d}\nu(x)}{c_{x,\epsilon}}\right|\mathrm{d}\mu_0(y) \\
&\leq \int_{\mathscr{X}}|\phi(y)|\Big(\int_{\mathscr{X}}\mathbb{1}_{\mathrm{d}(x,y)<\epsilon}\left|\frac{1}{c_{x^*,\epsilon}}-\frac{1}{c_{x,\epsilon}}\right|\mathrm{d}\mu(x)+ \\
&\quad \int_{\mathscr{X}}\mathbb{1}_{\mathrm{d}(x,y)<\epsilon}\left|\frac{1}{c_{x^*,\epsilon}}-\frac{1}{c_{x,\epsilon}}\right|\mathrm{d}\nu(x)+0\Big)\mathrm{d}\mu_0(y)
\end{aligned}$$

for $x^*\in\mathscr{X}$ fixed, since μ and ν coincide on balls with radius smaller than ϵ_0.

Moreover,

$$\begin{aligned}
&\int_{\mathscr{X}}|\phi(y)|\left(\int_{\mathscr{X}}\mathbb{1}_{\mathrm{d}(x,y)<\epsilon}\left|\frac{1}{c_{x^*,\epsilon}}-\frac{1}{c_{x,\epsilon}}\right|\mathrm{d}\mu(x)\right)\mathrm{d}\mu_0(y) \\
&\leq \sup_{x,y\in\mathscr{X}}\left|\frac{c_{x,\epsilon}}{c_{y,\epsilon}}-1\right|\int_{\mathscr{X}}\frac{1}{c_{x^*,\epsilon}}\int_{\mathscr{X}}\mathbb{1}_{\mathrm{d}(x,y)<\epsilon}|\phi(y)|\,\mathrm{d}\mu_0(y)\mathrm{d}\mu(x) \\
&\leq \sup_{x,y\in\mathscr{X}}\left|\frac{c_{x,\epsilon}}{c_{y,\epsilon}}-1\right|\|\phi\|_\infty\int_{\mathscr{X}}\frac{c_{x,\epsilon}}{c_{x^*,\epsilon}}\mathrm{d}\mu(x)\to 0,\ \epsilon\to 0,
\end{aligned}$$

since μ_0 is quasi-uniform.

Secondly, notice that

$$\int_{\mathscr{X}} K_\epsilon\phi(x)\mathrm{d}\mu(x)-\int_{\mathscr{X}}\phi(x)\mathrm{d}\mu(x)\to 0,\ \epsilon\to 0. \tag{S.6.4}$$

Indeed,

$$\begin{aligned}
\left|\int_{\mathscr{X}} K_\epsilon\phi(x)\mathrm{d}\mu(x)-\int_{\mathscr{X}}\phi(x)\mathrm{d}\mu(x)\right| &\leq \int_{\mathscr{X}}\left(\int_{\mathscr{X}}\mathbb{1}_{\mathrm{B}(x,\epsilon)}(z)\,|\phi(z)-\phi(x)|\,\frac{\mathrm{d}\mu_0(z)}{c_{x,\epsilon}}\right)\mathrm{d}\mu(x) \\
&\leq \omega_\epsilon(\phi).
\end{aligned}$$

where $\omega_\epsilon(\phi)=\sup_{x,y\in\mathscr{X},\mathrm{d}(x,y)<\epsilon}|\phi(x)-\phi(y)|$ converges to 0 when ϵ converges to 0 since ϕ is continuous on the compact $\mathscr{X}$, and thus uniformly continuous.

According to (S.6.3) and (S.6.4), also true for ν, it follows that $\int_{\mathscr{X}}\phi(x)\mathrm{d}\mu(x)=\int_{\mathscr{X}}\phi(x)\mathrm{d}\nu(x)$.

S.6.2. *Proofs for Section S.2 of the supplement.*

S.6.2.1. *Proof for Proposition S.2.1.* Let $x^*\in\mathscr{X}$ be such that $\mathrm{d}(x^*,\mathrm{Supp}(\mu))=\mathrm{d}_{\mathrm{H}}(\mathrm{Supp}(\mu),\mathscr{X})$, such a point x^* exists since $\mathrm{Supp}(\mu)$ is compact. Then,

$$\mathrm{d}_{\mu_0,h}(x^*)\leq\mathrm{d}_{\mu_0,h(\epsilon)}(x^*)<\epsilon\leq\mathrm{d}_{\mu,h}(x^*) \tag{S.6.5}$$

for every $0<h<h(\epsilon)$ since $\mu(\mathrm{B}(x^*,\epsilon))=0$ whereas $\mu_0(\mathrm{B}(x^*,r))>0$ for every $0<r<\epsilon$.

S.6.2.2. *Proof for Proposition S.2.2.* Let $l, h \in (0,1)$ and $x \in A_{l,h} := \{x \in \mathscr{X} \mid \mathrm{B}(x, r_{h(1-l)}) \subset \mathrm{Supp}(\mu_l)\}$. Then, since $\mathrm{Supp}(\mu_l)$ is closed, $\bar{B}(x, r_{h(1-l)}) \subset \mathrm{Supp}(\mu_l)$. Consequently, $\mu_l\left(\bar{B}(x, r_{h(1-l)})\right) = \frac{1}{1-l}\mu_0\left(\bar{B}(x, r_{h(1-l)})\right) \geq \frac{h(1-l)}{1-l} = h$, by definition of $r_{h(1-l)}$. Since $\mu_l(\bar{B}(x,r)) = \frac{1}{1-l}\mu_0(\bar{B}(x,r))$ for every $r \leq r_{h(1-l)}$, we deduce that $\delta_{\mu_l,m}(x) = \delta_{\mu_0,m(1-l)}(x) \leq r_{h(1-l)}$ for every $m < h$. Using (2.1), we conclude that $\mathrm{d}_{\mu_l,h}(x) = \mathrm{d}_{\mu_0,h(1-l)}(x) = \mathrm{d}_{h(1-l)}$ for every $x \in A_{l,h}$. So,

$$
\begin{aligned}
\mathcal{W}_1(s_h(\mu_l), s_h(\mu_0)) &= \int_{\mathbb{R}} \left| F^{-1}_{s_h(\mu_l)}(u) - F^{-1}_{s_h(\mu_0)}(u) \right| \mathrm{d}u \\
&= \int_{\mathbb{R}} \left| F^{-1}_{\mathrm{d}_{\mu_l,h}{}_{\#\mu_l}}(u) - \mathrm{d}_h \right| \mathrm{d}u \\
&\geq \mu_l(A_{l,h}) \left| \mathrm{d}_{h(1-l)} - \mathrm{d}_h \right| \\
&= \frac{1}{1-l}\mu_0(A_{l,h}) \left| \mathrm{d}_{h(1-l)} - \mathrm{d}_h \right|.
\end{aligned}
$$

The equivalence follows from the fact that $f : h \mapsto \mathrm{d}_h$ is differentiable at h (since it is expressed as an integral, cf. (2.1)), so $\left|\mathrm{d}_h - \mathrm{d}_{h(1-l)}\right| = l\,(hf'(h)) + O(l^2)$, so $|\mathrm{d}_{h(1-l)} - \mathrm{d}_h| \sim_{l\to 0} C_h l$ for some constant C_h. This constant C_h is non negative since f is non decreasing. Moreover, $C_h = 0$ if and only if f is locally constant around h, that is, if and only if $\mathscr{X}$ is discrete with $h < \frac{1}{|\mathscr{X}|}$.

S.6.2.3. *Proof of Proposition S.2.3.* Let $h \in \left[\frac{1}{N}, \frac{k+1}{N}\right]$. Let $\mu \neq \mu_0$. Let $x \in \mathscr{X}$ be such that $\mu(\{x\}) < \frac{1}{N}$. Then, $\mathrm{d}^2_{\mu,h}(x) \geq \frac{h-\mu(\{x\})}{h} d_2^2 > \frac{h-\frac{1}{N}}{h} d_2^2 = \mathrm{d}^2_{\mu_0,h}(x)$. Therefore, $s_h(\mu) \neq s_h(\mu_0)$.

The first lower bound follows from the expression:

$$
\mathcal{W}_1\left(s_h(\mu), s_h(\mu_0)\right) = \frac{1}{N} \sum_{x \in \mathscr{X}} \left| \mathrm{d}_{\mu_0,h}(x) - \mathrm{d}_{\mu,h}(x) \right|. \tag{S.6.6}
$$

The second lower bound follows from the inequality:

$$
\forall 0 \leq x \leq y \leq 1,\ \sqrt{1-x} - \sqrt{1-y} \geq \frac{1}{2}(y-x), \tag{S.6.7}
$$

and the fact that $\sum_{x\in\mathscr{X},\mu(\{x\})<\frac{1}{N}} \left(\frac{1}{N} - \mu(\{x\})\right) = \sum_{x\in\mathscr{X},\mu(\{x\})>\frac{1}{N}} \left(\mu(\{x\}) - \frac{1}{N}\right) = \frac{1}{2}\sum_{x\in\mathscr{X}} \left|\mu(\{x\}) - \frac{1}{N}\right|$.

The last two lower bounds are obtained by considering a coupling π of μ and μ_0 so that: $\pi(\{x\} \times \{x\}) = \min\left\{\frac{1}{N}, \mu(\{x\})\right\}$. Morally, for such a coupling, the mass $\sum_{x\in\mathscr{X},\mu(\{x\})>\frac{1}{N}} \left(\mu(\{x\}) - \frac{1}{N}\right)$ of μ is transported to another point, of a distance at most d_N, and the remaining mass is not transported.

S.6.2.4. *Proof of Propostion S.2.4.* Let $h \in \left[0, \frac{1}{N}\right)$, then, every probability measure $\mu \in \mathscr{P}(\mathscr{X})$ satisfying $\mu(\{x\}) \geq h$ for every $x \in \mathscr{X}$ is such that $s_h(\mu) = s_h(\mu_0) = \delta_0$.

S.6.2.5. *The circle* $\mathbb{S}^1$. Let $Rp_\theta \in \mathbb{S}^1_R$, then, $\mu_0(\mathrm{B}_{Rp_\theta,r}) = \frac{1}{2\pi} Leb([\theta - \frac{r}{R}, \theta + \frac{r}{R}]) = \frac{r}{\pi R}$ for every $r \in [0, R\pi]$. So, $\delta_{\mu_0,h} = \delta_{\mu_0,h}(x) = Rh\pi$. Moreover, $\mathrm{d}^2_{\mu_0,h}(x) = \frac{1}{h}\int_{l=0}^h \delta^2_{\mu_0,l}\mathrm{d}l = \frac{1}{h}\int_{l=0}^h R^2\pi^2 l^2 \mathrm{d}l = \frac{1}{3}\pi^2 R^2 h^2$.

So, $s_h(\mu_0) = \delta_{\mathrm{d}_h}$ with $\mathrm{d}_h = \sqrt{\frac{1}{3}}\pi R h$.

S.6.2.6. *The sphere* $\mathbb{S}^2$. Since $\mathbb{S}^2$ is homogeneous, for simplification, we consider the ball centered at $p_{0,0} = (1, 0, 0)$, $\mathrm{B}_{p_{0,0},r} = \{p_{\theta,\phi},\ \arccos(\cos\theta) \leq r\} = \{p_{\theta,\phi},\ \theta \leq r\}$, for $r \geq 0$.

S.6.2.6.1. For $h \leq \frac{1}{2}$. For $r \leq \frac{\pi}{2}$, $\mu_0(\mathrm{B}_{p_{0,0},r}) = \frac{1}{4\pi}\int_{\theta=0}^{r}\int_{\phi=0}^{2\pi}\sin\theta \mathrm{d}\theta\mathrm{d}\phi = \frac{1}{2}(1-\cos r)$. Therefore, $r_h := \delta_{\mu_0,h} = \arccos(1-2h)$ for $h \leq \frac{1}{2}$.

Then,

$$
\begin{aligned}
\mathrm{d}^2_{\mu_0,h}(x) &= \frac{1}{h}\int_0^h \delta^2_{\mu_0,l}\mathrm{d}l \\
&= \frac{1}{h}\int_0^h \arccos^2(1-2l)\mathrm{d}l \\
&= \frac{1}{2h}\int_{r=0}^{r_h} r^2\sin(r)\mathrm{d}r \\
&= \frac{1}{2h}\left(\left[-r^2\cos(r)\right]_0^{r_h} + \int_0^{r_h} 2r\cos(r)\, dr\right) \\
&= \frac{1}{2h}\left(\left[-r^2\cos(r)\right]_0^{r_h} + [2r\sin(r)]_0^{r_h} - \int_0^{r_h} 2\sin(r)\, dr\right) \\
&= \frac{1}{2h}\left(\left[-r^2\cos(r)\right]_0^{r_h} + [2r\sin(r)]_0^{r_h} + 2[\cos(r)]_0^{r_h}\right) \\
&= \frac{1}{2h}\left(-r_h^2\cos(r_h) + 2r_h\sin(r_h) + 2\cos(r_h) - 2\right),
\end{aligned}
$$

with the change of variables $r = \arccos(1-2l)$ on $[0, h]$.

S.6.2.6.2. For $h \geq \frac{1}{2}$. For $r \geq \frac{\pi}{2}$, $\mu_0(\mathrm{B}_{p_{0,0},r}) = 1 - \mu_0(\mathrm{B}_{p_{0,0},\pi-r}) = 1 - \frac{1}{2}(1-\cos(\pi-r))$. Therefore, $r_h := \delta_{\mu_0,h} = \pi - \arccos(-1+2h)$ for $h \geq \frac{1}{2}$.

Then,

$$
\begin{aligned}
\mathrm{d}^2_{\mu_0,h}(x) &= \frac{1}{2h}\int_{r=0}^{r_h} r^2\sin(r)\, dr \\
&= \frac{1}{2h}\left(-r_h^2\cos(r_h) + 2r_h\sin(r_h) + 2\cos(r_h) - 2\right),
\end{aligned}
$$

with the change of variables $r = \pi - \arccos(-1+2l)$ on $\left[\frac{\pi}{2}, h\right]$ and $r = \arccos(1-2l)$ on $\left[0, \frac{\pi}{2}\right]$.

S.6.2.7. *The flat torus* $\mathbb{T}^2$. The expression of $\delta_{\mu_0,h} = \sqrt{\frac{h}{\pi}}$ for $h \leq \frac{\pi}{4}$ is given by $h = \pi\delta^2_{\mu_0,h}$, the Lebesgue volume of the 2-dimensional ball with radius $\delta_{\mu_0,h}$. Then, $\mathrm{d}^2_{\mu_0,h}(x) = \frac{1}{h}\int_0^h \frac{l}{\pi}\mathrm{d}l = \frac{1}{2}\frac{h}{\pi}$.

S.6.2.8. *The Bolza surface* $\mathbb{B}$.

LEMMA S.6.1. *For every radius* $R \in [0, 1)$*, the* $\tilde{\mu}_0$*-measure of the Euclidean ball with radius* R*,* $\mathrm{B}_{\|\cdot\|,0,R}$*, is given by* $\tilde{\mu}_0(\mathrm{B}_{\|\cdot\|,0,R}) = \frac{4\pi R^2}{1-R^2}$.

PROOF. $\tilde{\mu}_0(\mathrm{B}_{\|\cdot\|,0,R}) = \int_0^R 8\pi \frac{r}{(1-r^2)^2}\mathrm{d}r = \frac{4\pi R^2}{1-R^2}$. □

LEMMA S.6.2. *The $\tilde{\mu}_0$-measure of the Bolza surface is given by:*

$$\mathscr{V} := \tilde{\mu}_0(\mathbb{B}) = \frac{4\pi}{\sqrt{2}-1} - 8\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{r_1^2(\theta) - \frac{1}{\sqrt{2}}}{(1 - r_1^2(\theta)(1-\frac{1}{\sqrt{2}}))}\, d\theta. \tag{S.6.8}$$

In particular, $\mathscr{V} \simeq 12.5664$.

PROOF. According to Section S.2.2, the Bolza surface domain is given by $\mathrm{B}_{0,R}\cap\bigcap_{j=1}^{8}\mathrm{B}_j^c$, with $R = 2^{-\frac{1}{4}}$.

The $\tilde{\mu}_0$-volume of the Bolza surface is thus given by $\mathscr{V} = \mathscr{V}_0 - 8\mathscr{V}_1$, with $\mathscr{V}_0 = \tilde{\mu}_0(\mathrm{B}_{0,R})$ and $\mathscr{V}_1 = \tilde{\mu}_0(\mathrm{B}_{0,R}\cap\mathrm{B}_j) = \tilde{\mu}_0(\mathrm{B}_{0,R}\cap\mathrm{B}_8)$ for any $j\in[\![1,8]\!]$, since the sets $\mathrm{B}_{0,R}\cap\mathrm{B}_j$ are mutually disjointed.

According to Lemma S.6.1, $\mathscr{V}_0 = \frac{4\pi}{\sqrt{2}-1}$.

Moreover,

$$\mathscr{V}_1 = \int_{\theta=-\frac{\pi}{8}}^{\frac{\pi}{8}}\int_{r=r(\theta)}^{2^{-\frac{1}{4}}} \frac{4r}{(1-r^2)^2}\mathrm{d}r\mathrm{d}\theta = 2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{\frac{1}{\sqrt{2}} - r^2(\theta)}{(1-r^2(\theta))\left(1-\frac{1}{\sqrt{2}}\right)}\mathrm{d}\theta, \tag{S.6.9}$$

with $r(\theta)$, the Euclidean distance between 0 and B_8 along the direction with angle θ. According to definitions of sinus and cosinus with ratios in right triangles and Pythagore theorem, $r(\theta)$ is given by:

$$r(\theta) = x_0\cos(\theta) - \sqrt{r_0^2 - \sin^2(\theta)x_0^2}. \tag{S.6.10}$$

A numerical approximation of $\mathscr{V}_1$ gives $\mathscr{V}_1 \simeq 2.2214$, so that $\mathscr{V} \simeq 12.5664$. □

Let $\delta_{\mu_0,l} = 2\,\mathrm{argth}(r_l)$ be the geodesic radius of the ball centered at 0 (with Euclidean radius r_l) that contains the proportion l of the mass of the uniform measure on the Bolza surface. According to Lemma S.6.1, $r_l = \sqrt{\frac{\mathscr{V}l}{4\pi+\mathscr{V}l}}$.

Then, setting $x = \sqrt{\frac{\mathscr{V}l}{4\pi}}$, and noting that $\mathrm{argth}\left(\frac{x^2}{1+x^2}\right) = \mathrm{argsh}(x)$, we get that:

$$\begin{aligned}
\mathrm{d}^2_{\mu_0,h}(0) &= \frac{1}{h}\int_{l=0}^{h}\delta^2_{\mu_0,l}\mathrm{d}l \\
&= \frac{1}{h}\int_{l=0}^{h}\left(2\,\mathrm{argth}\left(\sqrt{\frac{\mathscr{V}l}{4\pi+\mathscr{V}l}}\right)\right)^2\mathrm{d}l \\
&= \frac{4}{h}\int_{l=0}^{h}\mathrm{argsh}^2\left(\sqrt{\frac{\mathscr{V}l}{4\pi}}\right)\, dl \\
&= \frac{32\pi}{h\mathscr{V}}\int_{x=0}^{\sqrt{\frac{\mathscr{V}h}{4\pi}}}\mathrm{argsh}^2(x)x\, dx \\
&= \frac{32\pi}{h\mathscr{V}}\int_{y=\mathrm{argsh}(0)}^{\mathrm{argsh}\left(\sqrt{\frac{\mathscr{V}h}{4\pi}}\right)}\mathrm{sh}(y)y^2\,\mathrm{sh}'(y)\, dy
\end{aligned}$$

$$= \frac{8\pi}{h\mathscr{V}} \int_{\operatorname{argsh}(0)}^{\operatorname{argsh}\left(\sqrt{\frac{\mathscr{V}h}{4\pi}}\right)} (e^{2y} - e^{-2y}) y^2 \, dy$$

$$= \frac{4\pi}{h\mathscr{V}} \left[(2y^2+1)\operatorname{ch}(2y) - 2y\operatorname{sh}(2y)\right]_{\operatorname{argsh}(0)}^{\operatorname{argsh}\left(\sqrt{\frac{\mathscr{V}h}{4\pi}}\right)}$$

$$= \frac{4\pi}{h\mathscr{V}} \left(-1 + (2c_h^2+1)\operatorname{ch}(2c_h) - 2c_h\operatorname{sh}(2c_h)\right),$$

with $c_h = \operatorname{argsh}\left(\sqrt{\frac{\mathscr{V}h}{4\pi}}\right)$.

So, after simplification with the formula $\operatorname{ch}(2x) = 2\operatorname{sh}^2(x)+1$, $\operatorname{sh}(2x) = 2\operatorname{sh}(x)\operatorname{ch}(x) = 2\operatorname{sh}(x)\sqrt{1+\operatorname{sh}^2(x)}$:

$$\text{(S.6.11)} \quad \mathrm{d}^2_{\mu_0,h}(0) = 2\left(\operatorname{argsh}^2\left(\sqrt{\frac{h\mathscr{V}}{4\pi}}\right) + \left(\left(\operatorname{argsh}\left(\sqrt{\frac{h\mathscr{V}}{4\pi}}\right)\right)\sqrt{\frac{4\pi}{h\mathscr{V}}+1} - 1\right)^2\right).$$

S.6.2.9. *Proof of Proposition S.2.9.* Using triangular inequality, Proposition 2.1 in Brécheteau (2025), and the definition of $\bar{s}_{n,h,1}(\hat{\mathbb{p}}_n)$, we get that:

$$\mathcal{W}_1\left(s_h(\mu), \bar{s}_{n,h,1}(\hat{\mathbb{p}}_n)\right) \leq \mathbb{E}_\mu\left[\mathcal{W}_1(s_h(\mu), s_h(\hat{\boldsymbol{\mu}}_n))\right] + \mathbb{E}_\mu\left[\mathcal{W}_1(\bar{s}_{n,h,1}(\hat{\mathbb{p}}_n), s_h(\hat{\boldsymbol{\mu}}_n))\right]$$
$$\leq 2\mathbb{E}_\mu\left[\mathcal{W}_1(s_h(\mu), s_h(\hat{\boldsymbol{\mu}}_n))\right].$$

We conclude with Theorem 1.1 in Brécheteau (2025).

S.6.3. *Proofs for Section S.3 of the supplement.*

S.6.3.1. *Proof of Proposition S.4.1.* The distribution $\left(\tilde{\mu}_0\left(\mathrm{B}_{0,2^{-\frac{1}{4}}}\right)\right)^{-1}(\tilde{\mu}_0)_{|\mathrm{B}_{0,2^{-\frac{1}{4}}}}$ is the distribution of $R\exp(2i\pi V)$ with R and V independent, V uniform on $[0,1]$ and R with density $r \to (\sqrt{2}-1)\frac{2r}{(1-r^2)^2}$ with respect to the Lebesgue measure on $[0, 2^{-\frac{1}{4}}]$. The cumulative distribution function of R is given by $F_R : r \in [0, 2^{-\frac{1}{4}}] \mapsto (\sqrt{2}-1)\frac{r^2}{1-r^2}$. Therefore, the generalised inverse of its cumulative distribution function is given by $F_R^{-1} : u \in [0,1] \mapsto \sqrt{\frac{u}{\sqrt{2}-1+u}}$. We conclude with the fact that for U uniform on $[0,1]$, the cumulative distribution function of $F_R^{-1}(U)$ is F_R.

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